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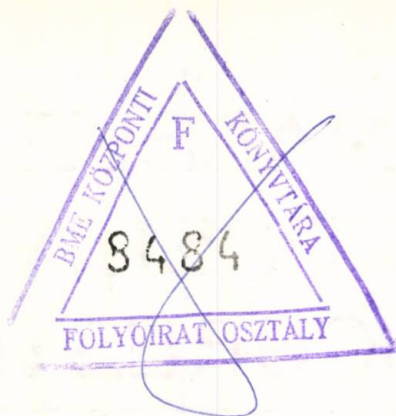
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CUPRINS • SUMMARY •

АРОЛСКА М.А., БАЙНОВ Д.Д., Двухточечная краевая задача для сингулярно возмущенной линейной системы дифференциальных уравнений	3
BAD M., Consistency of the distributions based on a sequential weighting process	9
BEJU A., Global properties for a nonlinear differential system	19
CIUCĂ I., Some considerations concerning the decoding of systematic linear codes	23
GHEORGHITĂ ȘT. I., On the motion of perfect fluids in the presence of an inhomogeneous porous body	39
GUPTA A. S., Effect of suspended particles on the Ekman boundary layer	45
IFTIMIE V., Remarque concernant l'approximation des solutions des équations fonctionnelles linéaires	53
MOCANU GH., A representation theorem for the space of Hermitian elements in a Banach algebra	57
PITICU M. E., Sur quelques résultats numériques pour les trajectoires de la masse nulle dans le système Terre-Lune	65
PHUNG T. Q., Construction d'une extension L/K de ramification triviale	75
POPESCU I., On computer generation of Burr and Pareto random variables	79
RAGAB F. M., SIMARY M. A., Definite integrals involving modified Bessel functions of the second kind	91
STATE L., On some asymptotic properties of the a posteriori distribution	103
TEODORESCU I. D., NICOLESCU L., Les espaces de Riemann ayant la forme de Ricci à courbure constante	133
Prof. Ionel Bucur (1930–1976)	143



ДВУХТОЧЕЧНАЯ КРАЕВАЯ ЗАДАЧА ДЛЯ СИНГУЛЯРНО ВОЗМУЩЕННОЙ СИСТЕМЫ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

М. АТ. АРОЛСКА, Д.Д. БАЙНОВ

Рассмотрим на отрезке $0 \leq t \leq 1$ линейную систему

$$\begin{aligned} (1) \quad y' &= A(t)y + B(t)z \\ \varepsilon z' &= C(t)y + D(t)z \end{aligned}$$

с краевым условием

$$\begin{aligned} (2) \quad L_1[y(0, \varepsilon), y(1, \varepsilon), \varepsilon] &= 0 \\ L_2[z(0, \varepsilon), z(1, \varepsilon), \varepsilon] &= 0 \end{aligned}$$

где $y, L_1 \in R_m, z, L_2 \in R_n, \varepsilon > 0$ — малый параметр.

Докажем, что при определенных условиях краевая задача (1)—(2) имеет решение, удовлетворяющее условиям

$$(3) \quad \lim_{\varepsilon \rightarrow 0} y(t, \varepsilon) = \bar{y}(t), \quad \lim_{\varepsilon \rightarrow 0} z(t, \varepsilon) = \bar{z}(t)$$

где $\bar{y}(t), \bar{z}(t)$ решение краевой задачи

$$\begin{aligned} (4) \quad \bar{y}'(t) &= A(t)\bar{y} + B(t)\bar{z} \\ 0 &= C(t)\bar{y} + D(t)\bar{z} \\ (5) \quad L_1[\bar{y}(0), \bar{y}(1), 0] &= 0 \end{aligned}$$

Отметим, что аналогичная задача, но с линейным краевым условием, была рассмотрена в [1].

Предположим, что выполнены следующие условия (А):

А1. Матрицы $A(t), B(t), C(t)$ и $D(t)$ непрерывны и $D(t)$ имеет собственные значения с отличными от нуля действительными частями для $0 \leq t \leq 1$.

А2. Вектор-функции L_1 и L_2 непрерывны и непрерывно дифференцируемы по своим аргументам.

Делая в замену переменных

$$(6) \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon T \\ -T & I_n + \varepsilon TS \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} = H(t, \varepsilon) \begin{pmatrix} V \\ W \end{pmatrix}$$

получим систему

$$(7) \quad \begin{aligned} v' &= [A(t) - B(t)T(t, \varepsilon)] V \\ \varepsilon w' &= [D(t) + \varepsilon T(t, \varepsilon) B(t)] W \end{aligned}$$

где $T(t, \varepsilon)$, $S(t, \varepsilon)$ — решение системы

$$(8) \quad \begin{aligned} \varepsilon T' &= D(t)T - \varepsilon TA(t) + \varepsilon TB(t)T - C(t) \\ \varepsilon S' &= \varepsilon[A(t) - B(t)T]S - S[D(t) - \varepsilon TB(t)] - B(t) \end{aligned}$$

Справедлива [1] следующая

ЛЕММА 1. Существует $\varepsilon_0 > 0$ такое, что решение $T(t, \varepsilon)$, $S(t, \varepsilon)$ системы (8) существует на сегменте $0 \leq t \leq 1$, равномерно ограничено при $0 \leq t \leq 1$, $0 < \varepsilon \leq \varepsilon_0$ и имеет место предельный переход

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} T(t, \varepsilon) &= D^{-1}(t)C(t) \\ \lim_{\varepsilon \rightarrow 0} S(t, \varepsilon) &= -B(t)D^{-1}(t) \end{aligned}$$

Отметим, что из условий (A) следует, что $D^{-1}(t)$ существует на сегменте $0 \leq t \leq 1$.

После замены переменных (6), краевые условия (2) принимают вид

$$(9) \quad \begin{aligned} L_1 \left[\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} H(0, \varepsilon) \begin{pmatrix} v(0, \varepsilon) \\ w(0, \varepsilon) \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} H(1, \varepsilon) \begin{pmatrix} v(1, \varepsilon) \\ w(1, \varepsilon) \end{pmatrix}, \varepsilon \right] &= 0 \\ L_2 \left[\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} H(0, \varepsilon) \begin{pmatrix} v(0, \varepsilon) \\ w(0, \varepsilon) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} H(1, \varepsilon) \begin{pmatrix} v(1, \varepsilon) \\ w(1, \varepsilon) \end{pmatrix}, \varepsilon \right] &= 0 \end{aligned}$$

Введем обозначение

$$P = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$$

где p ($0 \leq p \leq n$) — число собственных значений $D(t)$, которые имеют отрицательные действительные части.

Решение задачи (7)–(9) будем искать в виде

$$(10) \quad \begin{aligned} v(t, \varepsilon) &= V(t, \varepsilon) \alpha_1(\varepsilon) \\ w(t, \varepsilon) &= [W(t, \varepsilon) P W^{-1}(0, \varepsilon) + \\ &+ W(t, \varepsilon)(I_n - P)W^{-1}(1, \varepsilon)] \alpha_2(\varepsilon) \end{aligned}$$

где $V(t, \varepsilon)$, $W(t, \varepsilon)$ — фундаментальные матрицы систем

$$(11) \quad V' = (A - BT)V$$

$$(12) \quad \varepsilon W' = (D - \varepsilon TB)W$$

Имеет место следующая

ЛЕММА 2. Существует независимые от ε постоянные K , μ и σ и постоянная $\varepsilon_0 > 0$ такие, что при $0 < \varepsilon \leq \varepsilon_0$ справедливы неравенства

$$||W(t, \varepsilon)PW^{-1}(s, \varepsilon)|| \leq Ke^{-\frac{\mu(t-s)}{\varepsilon}} \quad 0 \leq s \leq t \leq 1$$

$$||W(t, \varepsilon)(I_n - P)W^{-1}(s, \varepsilon)|| \leq Ke^{-\frac{\mu(s-t)}{\varepsilon}}, \quad 0 \leq s \leq t \leq 1$$

$$||V(t, \varepsilon)V^{-1}(s, \varepsilon)|| \leq e^{\sigma|t-s|}, \quad 0 \leq t, s \leq 1$$

Подставляя (10) в краевые условия (9), получим

$$(13) \quad L_1 \left[\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} H(0, \varepsilon) \Delta(0, \varepsilon) \begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \times \right. \\ \left. \times H(1, \varepsilon) \Delta(1, \varepsilon) \begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix}, \varepsilon \right] = 0$$

$$L_2 \left[\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} H(0, \varepsilon) \Delta(0, \varepsilon) \begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} \times \right. \\ \left. \times H(1, \varepsilon) \Delta(1, \varepsilon) \begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon) \end{pmatrix}, \varepsilon \right] = 0$$

где

$$\Delta(t, \varepsilon) = \begin{pmatrix} V(t, \varepsilon) & 0 \\ 0 & W(t, \varepsilon)PW^{-1}(0, \varepsilon) + W(t, \varepsilon)(I_n - P)W^{-1}(1, \varepsilon) \end{pmatrix}$$

Предположим, что выполнено следующее условие (Б): При $\varepsilon = 0$ система (13) разрешима относительно α_1 , α_2 и определитель

$$\Delta_0^0 = \frac{D(L_1, L_2)}{D(\alpha_1, \alpha_2)} \bigg|_{\substack{\alpha_1 = \alpha_1^0 \\ \alpha_2 = \alpha_2^0 \\ \varepsilon = 0}} \neq 0$$

где α_1^0 , α_2^0 — решение системы (13) при $\varepsilon = 0$.

Тогда из теоремы о неявных функций следует, что при $0 < \varepsilon \leq \varepsilon_0$ существуют функции $\alpha_1^0(\varepsilon)$ и $\alpha_2^0(\varepsilon)$, которые удовлетворяют системе (13) и краевая задача (7) — (9) имеет решение

$$(14) \quad \begin{aligned} v(t, \varepsilon) &= V(t, \varepsilon) \alpha_1^0(\varepsilon) \\ w(t, \varepsilon) &= [W(t, \varepsilon) P W^{-1}(0, \varepsilon) + \\ &+ W(t, \varepsilon)(I_n - P) W^{-1}(1, \varepsilon)] \alpha_2^0(\varepsilon) \end{aligned}$$

Отметим, что из леммы 2 следует, что

$$(15) \quad \begin{aligned} \lim_{\varepsilon \rightarrow 0} v(t, \varepsilon) &= V(t) \alpha_1^0, \quad 0 \leq t \leq 1 \\ \lim_{\varepsilon \rightarrow 0} w(t, \varepsilon) &= 0, \quad 0 < t < 1 \end{aligned}$$

где $V(t) = \lim_{\varepsilon \rightarrow 0} V(t, \varepsilon)$

Введем обозначение

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \lim_{\varepsilon \rightarrow 0} \begin{pmatrix} v(t, \varepsilon) \\ w(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} V(t) \alpha_1^0 \\ 0 \end{pmatrix}, \quad 0 < t < 1$$

Легко видно, что $x(t)$ удовлетворяет вырожденной системе, которая получается из (7) при $\varepsilon = 0$:

$$(16) \quad \begin{aligned} x_1' &= [S(t) - B(t)D^{-1}(t) C(t)]x_1 \\ 0 &= D_2(t)x_2 \end{aligned}$$

и краевом условии

$$\begin{aligned} L_1 \left[\begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} H(0, 0) x(0), \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} H(1, 0) x(1), 0 \right] = \\ = L_1[\bar{y}(0), \bar{y}(1), 0] = 0 \end{aligned}$$

Следовательно справедлива следующая

ТЕОРЕМА. Пусть выполнены условия (А) и (Б), Тогда существует постоянная $\varepsilon_0 > 0$ такая, что при $0 < \varepsilon \leq \varepsilon_0$ решение краевой задачи (1) — (2) существует на сегменте $0 \leq t \leq 1$ представимо в виде

$$\begin{pmatrix} y(t, \varepsilon) \\ z(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon S(t, \varepsilon) \\ -T(t, \varepsilon) I_n + \varepsilon T(t, \varepsilon) S(t, \varepsilon) \end{pmatrix} \begin{pmatrix} v(t, \varepsilon) \\ w(t, \varepsilon) \end{pmatrix}$$

где матрица $\begin{pmatrix} v(t, \varepsilon) \\ w(t, \varepsilon) \end{pmatrix}$ выражается через (14) и для $0 < t < 1$ имеет место предельный переход

$$\lim_{\varepsilon \rightarrow 0} \begin{pmatrix} y(t, \varepsilon) \\ z(t, \varepsilon) \end{pmatrix} = \begin{pmatrix} \bar{y}(t) \\ \bar{z}(t) \end{pmatrix}$$

где

$$\bar{y}'(t) = A(t)\bar{y} + B(t)\bar{z}$$

$$0 = C(t)\bar{y} + D(t)\bar{z}$$

и

$$L_1(\bar{y}(0), \bar{y}(1), 0) = 0$$

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CONSISTENCY OF THE DISTRIBUTIONS BASED ON A SEQUENTIAL WEIGHTING PROCESS

MONICA BAD

Many authors studied the limiting behaviour of Bayes' posterior distributions and of Bayes' estimators, proving the asymptotic consistency in different conditions.

The paper establishes the conditions under which a range of posterior distributions based on a sequential weighting process with independent observations, converges to a degenerated distribution. The obtained theorem is used in order to obtain the asymptotic consistency of the parametric estimators based on the same weighting process.

Let $X, X_1, X_2, \dots$ be random independent and identically distributed variables on a probability space, let S be the selection space corresponding to X and $\mathcal{B}$ an σ -algebra on S .

We suppose that the X -variables have a distribution depending on a parameter $\theta \in \Theta$. Let $\mathcal{A}$ be an σ -algebra on Θ .

As the real value of the parameter is fixed but unknown, we shall consider that $P: \mathcal{B} \rightarrow [0, 1]$ is the real distribution of the „observations”.

Let $\lambda: \mathcal{A} \rightarrow [0, 1]$ be an a priori probability on Θ , and $\{u_k(x, \theta)\}_k$ a range of „weights”

$u_k(x, \theta): S \times \Theta \rightarrow R_{\geq 0}$, $\mathcal{B} \times \mathcal{A}$ -measurable for any k .

The posterior distribution of θ , after $(x_1, \dots, x_n)$ was observed will be given by a sequential weighting process (see [3]):

$$\mu_n(A) = \frac{\int_A \prod_{k=1}^n u_k(x_k, \theta) d\lambda(\theta)}{\int_{\Theta} \prod_{k=1}^n u_k(x_k, \theta) d\lambda(\theta)} \quad (\forall) A \in \mathcal{A} \quad (1)$$

Notation:

We note: $l_k(x, \theta) = \log u_k(x, \theta)$

$$\bar{l}_n(\theta) = \bar{l}_n(x_1 \dots x_n, \theta) = \frac{1}{n} \sum_{k=1}^n l_k(x_k, \theta)$$

$$\eta_k(\theta) = E(l_k(x, \theta)) \text{ (expectation according to } P)$$

$$\eta(\theta) = \lim_{k \rightarrow \infty} \eta_k(\theta) \text{ (we shall suppose it exists } \lambda\text{-a.s.)}$$

$$\eta^* = \sup_{\theta \in \Theta} \eta(\theta)$$

$$A_\delta = \{\theta | \eta^* - \delta \leq \eta(\theta) \leq \eta^*\} \quad \text{for any } \delta > 0$$

For $v: \Theta \rightarrow R$, $\mathcal{A}$ -measurable and for an $A \in \mathcal{A}$ we note: $||v||_A^{(x)} = \sup_{\theta \in A} |v(\theta)|$

$$||v||_A^{(n)} = \left[\int_A |v|^n d\lambda \right]^{1/n}$$

When $A = \Theta$, we shall omit the Θ -index in notation.

Assumptions: Let us assume:

(A.1) $u_k(x, \theta)$ is $\mathcal{B} \times \mathcal{A}$ -measurable

(A.2) $\lambda\{\theta | P\{x | u_k(x, \theta) > 0\} = 1\} > 0$ for any k in N , where the measurability is ensured by Fubini's theorem.

(A.3) (a) $\lambda\{\theta | E(l_k(x, \theta)) < \infty\} = 1 \text{ } (\forall) k \in N$

(b) $\lambda\{\theta | D^2(l_k(x, \theta)) < \infty\} = 1 \text{ } (\forall) k \in N$

(c) $\lambda \left\{ \theta \mid \sum_{k=1}^{\infty} \frac{D^2(l_k(x, \theta))}{k^2} < \infty \right\} = 1$

(A.4) there exists $\eta: \Theta \rightarrow R$, so that

$$\eta_k(\theta) \xrightarrow[k \rightarrow \infty]{} \eta(\theta), \quad \lambda - \text{a.s.}$$

Remark (1)

(A.2) ensures the fact that $\mu_n(A)$ is well defined almost sure, that means the denominator in (1) is not null P-a.s.

Remark (2)

If we possibly wish that the denominator in (1) is finite, we shall have to impose some new conditions.

In this purpose, let us note:

$$Z_n = \int_{\Theta} \prod_{k=1}^n u_k(x_k, \theta) d\lambda(\theta) = \int_{\Theta} \exp\{n \bar{l}_n(\theta)\} d\lambda(\theta)$$

$$Z_0 = \lambda(\Theta) = 1$$

$$t = \begin{cases} \text{the first } n \text{ for which } Z_n < \infty \\ \infty, \text{ when } n \text{ doesn't exist.} \end{cases}$$

$$t' = \begin{cases} \text{the last } n \text{ for which } Z_n = \infty \\ 0, \text{ when all } Z_n < \infty \\ \infty, \text{ when all } Z_n = \infty \end{cases}$$

We call: λ „becomes proper” if $P(t < \infty) = 1$

λ „remains proper” if $P(t' < \infty) = 1$

We see that $t \leq t' + 1$

Remark (3)

The relation (1) may be rewritten as :

$$\mu_n(A) = \frac{\int_A \exp \{n \bar{l}_n\} d\lambda}{\int_{\Theta} \exp \{n \bar{l}_n\} d\lambda}, \text{ or}$$

$$\mu_n(A) = \left[\frac{||\exp \bar{l}_n||_A^{(n)}}{||\exp \bar{l}_n||^{(n)}} \right]^n (\forall) A \in \mathcal{A} \quad (2)$$

We shall be interested in the asymptotic behaviour of $||\exp \bar{l}_n||_A^{(n)}$, $A \in \mathcal{A}$.

Lemma (1)

When we suppose (A.3), (A.4) to be true, we get :

$$P \left[\bar{l}_n \xrightarrow[n \rightarrow \infty]{\lambda - a.s.} \eta \right] = 1$$

Proof :

$\{\bar{l}_n(\theta)\}_n$ form a range of successive expectations of the random independent and notidentically distributed variables $\{l_k(X, \theta)\}_k$.

Then, by (A.3) (c), we can apply the strong law of large numbers and we get :

$$|\bar{l}_n(\theta) - \frac{1}{n} \sum_{k=1}^n \eta_k(\theta)| \xrightarrow[n \rightarrow \infty]{P - a.s.} 0, \lambda - a.s.$$

But, by (A.4) $\gamma_k(\theta) \xrightarrow[k \rightarrow \infty]{\lambda-a.s.} \gamma(\theta)$ and so does the Cesaro range :

$$\frac{1}{n} \sum_{k=1}^n \gamma_k(\theta) \xrightarrow[n \rightarrow \infty]{\lambda-a.s.} \gamma(\theta)$$

$$\text{Then } P \left[|\bar{l}_n(\theta) - \gamma(\theta)| \xrightarrow[n \rightarrow \infty]{\lambda-a.s.} 0 \right] = 1.$$

q.e.d.

Further we shall use Lemma (2) and Corolar (1) form [2], but we don't resume their proofs.

Lemma (2)

If $v_n \xrightarrow{\lambda-a.s.} v$, then

$$\liminf_{n \rightarrow \infty} ||v_n||^{(n)} \geq ||v||^{(\infty)}$$

Corolar (1)

If $v, v_1, v_2, \dots$ are random functions, $v = v(x, \theta)$, $v_i = v_i(x, \theta)$ and if

$$P \left[v_n \xrightarrow{\lambda-a.s.} v \right] = 1,$$

$$\text{then } P[\liminf_{n \rightarrow \infty} ||v||^{(n)} \geq ||v||^{(\infty)}] = 1.$$

Lemma (3)

If λ „becomes proper” and if for an $A \in \mathcal{A}$ we have :

$$P[\limsup_{n \rightarrow \infty} (\sup_{\theta \in A} \bar{l}_n(\theta)) \leq a] = 1, \text{ then } P[\limsup_{n \rightarrow \infty} ||\exp \bar{l}_n||_A^{(n)} \leq e^a] = 1$$

Proof :

Let t be the first m for which $Z_m < \infty$

By hypothesis we have $P(t < \infty) = 1$

$$\begin{aligned} ||\exp \bar{l}_n||_A^{(n)} &= \left[\int_A |\exp(n \bar{l}_n)| d\lambda \right]^{1/n} = \\ &= \left[\int_A \exp(t \bar{l}_t) d\lambda \right]^{1/n} \cdot \left[\int_A \exp \left(n \cdot \frac{1}{n} \sum_{k=t+1}^n l_k(x_k, \theta) \right) d\lambda \right]^{1/n} \end{aligned}$$

If we note

$$\bar{l}'_n = \frac{1}{n} \sum_{k=l+1}^n l_k(x_k, \theta), \text{ we get :}$$

$$||\exp \bar{l}_n||_{\mathcal{A}}^{(n)} = \left[\int_{\mathcal{A}} \exp(t \bar{l}_t) d\lambda \right]^{1/n} \cdot ||\exp \bar{l}'_n||_{\mathcal{A}}^{(n)}$$

And so, on $\{t < n\}$ we get :

$$||\exp \bar{l}_n||_{\mathcal{A}}^{(n)} \leq \left[\int_{\mathcal{A}} \exp(t \bar{l}_t) d\lambda \right]^{1/n} \cdot \exp \left(\sup_{\theta \in \mathcal{A}} \bar{l}'_n(\theta) \right), P - \text{a.s.}$$

But the behaviour of $\bar{l}'_n$, conditional by t , is the same with that of $\bar{l}_n$, that means :

$$P[\limsup_{n \rightarrow \infty} \left(\sup_{\theta \in \mathcal{A}} \bar{l}'_n(\theta) \right) \leq a | t] = 1$$

It follows :

$$P[\limsup_{n \rightarrow \infty} \left(\exp \left(\sup_{\theta \in \mathcal{A}} \bar{l}'_n(\theta) \right) \right) \leq e^a | t] = 1$$

$$\text{But as } \limsup_{n \rightarrow \infty} \left[\int_{\mathcal{A}} \exp(t \bar{l}_t) d\lambda \right]^{1/n} = 1,$$

$$\text{we get : } P(\limsup_{n \rightarrow \infty} ||\exp \bar{l}_n||_{\mathcal{A}}^{(n)} \leq e^a) = 1$$

q.e.d.

Theorem (1)

Let $A \in \mathcal{A}$ which verifies the following conditions :

- (i) λ „becomes proper”
- (ii) (A.1) — (A.4) are true, and $\eta^* < \infty$
- (iii) $P[\limsup_{n \rightarrow \infty} \left(\sup_{\theta \in \mathcal{A}} \bar{l}(\theta) \right) < \eta^* - \gamma] = 1$ for $\gamma > 0$

$$\text{Then } P[\mu_n(A) \xrightarrow[n \rightarrow \infty]{} 0] = 1$$

Proof :

$$\text{By Lemma (1) we have : } P[\bar{l}_n(\theta) \xrightarrow{\lambda - \text{a.s.}} \eta(\theta)] = 1$$

$$\text{By Lemma (2), and as } P[\exp \bar{l}_n(\theta) \xrightarrow{\lambda - \text{a.s.}} \eta(\theta)] = 1$$

$$\text{we obtain } P[\liminf_{n \rightarrow \infty} ||\exp \bar{l}_n||_{\mathcal{A}}^{(n)} \geq ||\exp \eta||_{\mathcal{A}}^{(\infty)} = \exp \eta^*] = 1$$

By Lemma (3) and by (iii) it follows :

$$P[\limsup_{n \rightarrow \infty} ||\exp \bar{l}_n||_A^{(n)} < \exp (\eta^* - \gamma)] = 1$$

As $[\mu_n(A)]^{1/n} = ||\exp \bar{l}_n||_A^{(n)} / ||\exp \bar{l}_n||^{(n)}$, it follows :

$$P[\limsup_{n \rightarrow \infty} (\mu_n(A))^{1/n} < e^{-\gamma}] = 1$$

and we get $P[\lim_{n \rightarrow \infty} \mu_n(A) = 0] = 1$

q.e.d

Further we'll take $A = A'_\delta = \Theta - A$,

in order to obtain $P[\mu_n(A) \xrightarrow[n \rightarrow \infty]{} 1] = 1 \quad (\forall) \delta > 0$

The conditions (i), (ii) in Theorem (1) don't depend on A, so that it remains to establish which are the conditions for

$$P[\limsup_{n \rightarrow \infty} (\sup_{\theta \in A'_\delta} \bar{l}_n(\theta) < \eta^* - \gamma) = 1 \text{ for a } \gamma > 0]$$

We make some new assumptions :

(A.5) $\{\eta_k(\cdot)\}_k$ is a nonincreasing range of equicontinuous functions of θ
 (A.6) there exists $p \in N$, so that for any $r \in R$ there exists a co-compact set D (that means D and $D' = \Theta - D$ are compact) with property :

$$E(\sup_{\theta \in D} \bar{l}_n(\theta)) < r.$$

Lemma (4)

If (A.1) — (A.6) occur, we have :

- (a) $\eta(\cdot)$ is continuous λ a.s.
- (b) η^* is finite
- (c) A_δ is a compact set for any $\delta > 0$

Proof :

(a) As $\eta_k(\theta) \rightarrow \eta(\theta)$ λ — a.s. and as $\{\eta_k(\cdot)\}_k$ is an equicontinuous family, it follows that $\eta(\theta)$ is continuous λ — a.s.

$$(b) \quad \eta^* = \sup_{\theta \in \Theta} \eta(\theta)$$

By (A.6), there exists p so that for any $r < \infty$ there exists a co-compact set D so that

$$E(\sup_{\theta \in D} \bar{l}_p(\theta)) < r < \infty$$

For $\theta \in D$, $|\eta(\theta)| \leq |\eta_p(\theta)| \leq |E(\sup_{\theta \in D} l_p(x, \theta))| \leq$

$$\leq |E(\sup_{\theta \in D} \bar{l}_p(\theta))| < \infty, \text{ by (A.5)}$$

For $\theta \in D'$, $|\eta(\theta)| \leq |\eta_p(\theta)| < \infty$, by (A.3)

Then $|\eta^*| < \infty$

(c) $A_\delta = \eta^{-1}[\eta^* - \delta, \eta^*]$ is closed

By (A.6), there exists p so that, for $r = \eta^* - \delta$ there exists D co-compact so that $E(\sup_{\theta \in D} \bar{l}_p(\theta)) < \eta^* - \delta$

For $\theta \in D$, $\eta(\theta) \leq \eta_p(\theta) \leq E(\sup_{\theta \in D} l_p(x, \theta)) \leq E(\sup_{\theta \in D} \bar{l}_p(\theta)) < \eta^* - \delta$

Then $A_\delta \subset D'$, A_δ closed, D' compact.

It follows A_δ compact.

q.e.d.

Theorem (2)

Let us suppose that :

(i) λ „becomes proper”

(ii) (A.1) – (A.6) are true.

$$\text{Then } P[\mu_n(A'_\delta) \xrightarrow{n \rightarrow \infty} 0] = 1 \quad (\forall) \delta > 0$$

$$\text{and } P[\mu_n(A_\delta) \xrightarrow{n \rightarrow \infty} 1] = 1 \quad (\forall) \delta > 0$$

Proof :

We verify the condition (iii) in Theorem (1), that means we verify that for any $\delta > 0$ there exists $\gamma > 0$ so that :

$$P[\limsup_{n \rightarrow \infty} (\sup_{\theta \in A'_\delta} \bar{l}_n(\theta)) < \eta^* - \gamma] = 1$$

Let be $\delta > 0$ arbitrary and let be $A_\delta = \{\theta | \eta^* - \delta \leq \eta(\theta) \leq \eta^*\}$

By (A.6), there exists p so that for $r = \eta^* - \delta$ there exists D co-compact so that :

$$E(\sup_{\theta \in D} \bar{l}_p(\theta)) < \eta^* - \delta$$

We notice that :

when $\theta \in D$, $\eta(\theta) \leq \eta_p(\theta) \leq E(\sup_{\theta \in D} \bar{l}_p(\theta)) < \eta^* - \delta$

and then $D \subset A'_\delta$, $A'_\delta = D \cup (D' \setminus A_\delta)$

It follows :

$$\sup_{\theta \in A'_\delta} \bar{l}_n(\theta) = \max \left\{ \sup_{\theta \in D} \bar{l}_n(\theta), \sup_{\theta \in D' \setminus A_\delta} \bar{l}_n(\theta) \right\} \quad (3)$$

We prove that both terms are bounded.

As D' compact, A_δ closed, it follows that there exists $\theta' \in \overline{D' \setminus A_\delta}$ so that :

$$\bar{l}_n(\theta') = \sup_{\theta \in D' \setminus A_\delta} \bar{l}_n(\theta)$$

But $\bar{l}_n(\theta') \xrightarrow[n \rightarrow \infty]{} \eta(\theta')$, P - a.s. and, for $\theta' \in D'$, $\eta(\theta') < \eta^* - \delta$

Then $\sup_{\theta \in D' \setminus A_\delta} \bar{l}_n(\theta) < \eta^* - \delta$ P - a.s., for a n sufficient

large (4)

We prove that for $n \geq p$, the first term is bounded too.

Let α be a selection of volume p from $\{1, \dots, n\}$

$$\frac{1}{p} \sum_{k=1}^p l_k(x_{\alpha k}, \theta) = \bar{l}_p(x_{\alpha 1}, \dots, x_{\alpha p}, \theta) \leq \sup_{\theta \in D} \bar{l}_p(x_{\alpha 1}, \dots, x_{\alpha p}, \theta)$$

Summing up for all the C_n^p values of α , we get :

$$\bar{l}_n(\theta) \leq \frac{1}{C_n^p} \sum_{\alpha} \sup_{\theta \in D} \bar{l}_p(x_{\alpha 1}, \dots, x_{\alpha p}, \theta) = \hat{l}_n(D)$$

So, for $\theta \in D$, we have proved that $\bar{l}_n(\theta)$ is bounded by a random variable $\hat{l}_n(D)$ which doesn't depend on θ .

But we can apply the convergency theorem for martingales for $\{\hat{l}_n(D)\}_n$.

We shall consider $\Omega = S^{(\infty)}$, a probability on Ω given by P and denoted P too and $\mathcal{F}_p \subseteq \mathcal{F}_{p+1} \subseteq \dots$, where $\mathcal{F}_k$ is σ -algebra $\mathcal{B}(\hat{l}_k(D))$ for any $k = p, p+1, \dots$

Let us note $z = \sup_{\theta \in D} \bar{l}_p(x_{\alpha 1}, \dots, x_{\alpha p}, \theta)$, (for each α)

Then $E(z | \mathcal{F}_n) = E(\sup_{\theta \in D} \bar{l}_p(\theta) | \mathcal{F}_n)$ is a martingale sequence and

$$\lim_{n \rightarrow \infty} E(\sup_{\theta \in D} \bar{l}_p(\theta) | \mathcal{F}_n) = E(\sup_{\theta \in D} \bar{l}_p(\theta) | \mathcal{F}_\infty), \quad P \text{ a.s.}$$

We notice that, when summing-up for all the values of α we get :

$$\begin{aligned} C_n^p E(\sup_{\theta} \bar{l}_p(\theta) | \mathcal{F}_n) &= E\left(\sum_{\alpha} \sup_{\theta \in D} \bar{l}_p(x_{\alpha 1}, \dots, x_{\alpha p}, \theta) | \mathcal{F}_n\right) = \\ &= E(C_n^p \hat{l}_n(D) | \mathcal{F}_n) = C_n^p \hat{l}_n(D) \end{aligned}$$

$$\text{So, } \hat{l}_n(D) = E(\sup_{\theta \in D} \bar{l}_p(\theta) | \mathcal{F}_n) \text{ and then}$$

$$\lim_{n \rightarrow \infty} \hat{l}_n(D) = E(\sup_{\theta \in D} \bar{l}_p(\theta) | \mathcal{F}_{\infty}) = E(\sup_{\theta \in D} \bar{l}_p(\theta)) \text{ P.a.s.}$$

It follows :

$$\lim_{n \rightarrow \infty} \hat{l}_n(D) < \eta^* - \delta, \text{ P.a.s. and then}$$

$$\limsup_{n \rightarrow \infty} (\sup_{\theta \in D} \bar{l}_n(\theta)) < \eta^* - \delta, \text{ P.a.s.} \quad (5)$$

By (3), (4), (5) it follows :

$$\limsup_{n \rightarrow \infty} (\sup_{\theta \in A_{\delta}} \bar{l}_n(\theta)) < \eta^* - \delta \text{ P.a.s.}$$

and this is the relation we were looking for with $\gamma = \delta$.

Then we can apply Theorem (1) and we get :

$$P[\mu_n(A_{\delta}) \xrightarrow{n \rightarrow \infty} 1] = 1 \quad \text{q.e.d.}$$

Theorem (3)

If Θ is a Hausdorff topological space and the open sets belong to $\mathcal{A}$, if there exists $\theta^* \in \Theta$ so that $\{A_{\delta} | \delta > 0\}$ forms a base for the θ^* -neighbourhoods and if the conditions of Theorem (2) are verified, then the μ_n distributions based on the sequential weighting process converge P.a.s. to a degenerated distribution, concentrated in $\{\theta^*\}$

Proof :

By hypothesis, for any V θ^* -neighbourhood there exists $A_{1/m} \subset V$

But as $P[\mu_n(A_{1/m}) \xrightarrow{n \rightarrow \infty} 1] = 1$, for any m , it follows :

$$P[\mu_n(V) \xrightarrow{n \rightarrow \infty} 1] = 1$$

$$\text{and then } \mu_n \xrightarrow{P\text{-a.s.}} \chi_{\{\theta^*\}} \quad \text{q.e.d.}$$



Corolar (2)

Let $t_n = \int_{\Theta} \theta d\mu_n(\theta)$ be a range of estimators for the θ -parameter, with μ_n given by (1)

If the conditions of Theorem (3) are satisfied, then $(t_n)_n$ forms a strong consistent range of estimators for θ^* , that is

$$P(t_n \rightarrow \theta^*) = 1$$

Remark (4)

If $u_k(x, \theta) = f(x|\theta)$ for any k in N , where $f(x|\theta)$ is a probability density for X with respect to a σ -finite measure, then

$$\mu_n(A) = \frac{\int_A \prod_{k=1}^n f(x_k | \theta) d\lambda(\theta)}{\int_{\Theta} \prod_{k=1}^n f(x_k | \theta) d\lambda(\theta)}$$

In order to obtain all previous results, we have to assume that (A.1), (A.2), (A.3), (A.6), are true and that $f(x|\theta)$ is continuous P -a.s., as a θ -function (see [1], [2]).

The assumptions (A.4), (A.5) are verified, while $\{\eta_k(\theta)\}_k$ is a range of constants $\eta_k(\theta) = \eta(\theta) = E(\log f(x|\theta))$

In this case $\eta(\theta)$ is an entropy and when $P(x)$ is $P_{\theta^*}(x)$, we get $A_{\delta} \xrightarrow{\delta \rightarrow 0} \{\theta^*\}$ and θ^* is the true value of the parameter.

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GLOBAL PROPERTIES FOR A NONLINEAR DIFFERENTIAL SYSTEM

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§1. Introduction

The principal aim of this paper is to give some global properties for a nonlinear differential system of (3.1) type.

So, a vectorial product for „n” vectors in R^{n+1} is defined and its properties are studied. By means of this vectorial product we prove for the system (3.1) that the trajectory (the curve-solution) is in a plane. The global characteristics of this plane are defined by Cauchy initial dates (3.2).

In the last part of this paper the connection with some differential system of Newtonian type are presented.

§2. Vectorial product in R^{n+1}

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n vectors in R^{n+1} with the components $\mathbf{v}_i = (v_{i1}, v_{i2}, \dots, v_{in}, v_{in+1})$ respectively.

We define the mapping $f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n; \cdot) : R^{n+1} \rightarrow R$ by the relation

$$(2.1) \quad f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n; \mathbf{x}) = \det (\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n), \quad \mathbf{x} \in R^{n+1}, \quad \text{where}$$

$$(2.2) \quad \det (\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = \begin{vmatrix} x_1 & x_2 & \dots & x_{n+1} \\ v_{11} & v_{12} & \dots & v_{1n+1} \\ \cdot & \cdot & \cdot & \cdot \\ v_{n1} & v_{n2} & \dots & v_{nn+1} \end{vmatrix}$$

We see that the mapping f is linear and then we have

$$(2.3) \quad f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n; \mathbf{x}) = \sum_{i=1}^{n+1} a_i x_i,$$

where

$$(2.4) \quad a_i = f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n; \mathbf{e}_i),$$

and $\{\mathbf{e}_i\}$ denotes the canonical basis in R^{n+1} .

So we associate to $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, by (2.3) a vector $\mathbf{a} \in R^{n+1}$ with the components (2.4).

Definition 2.1 The vector $\mathbf{a}$ will be named the *vectorial product* of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, and this will be written $\mathbf{a} = \mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_n$.

Remark 2.1. The vectorial product is a mapping from $(R^{n+1})^n$ in to R^{n+1} and generalizes the classical vectorial product in R^{2+1} .

This vectorial product has the following principal properties :

(i) it is total antisymmetric, that is

$$(2.5) \quad \mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_i \times \mathbf{v}_{i+1} \times \dots \times \mathbf{v}_{j-1} \times \mathbf{v}_j \times \dots \times \mathbf{v}_n = \\ = -\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_j \times \mathbf{v}_{i+1} \times \dots \times \mathbf{v}_{j-1} \times \mathbf{v}_i \times \dots \times \mathbf{v}_n;$$

(ii) for $\lambda \in R$ and $i = 1, 2, \dots, n$ we have

$$(2.6) \quad \mathbf{v}_1 \times \mathbf{v}_2 \times \dots (\lambda \mathbf{v}_i) \times \dots \times \mathbf{v}_n = \lambda (\mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_i \times \dots \times \mathbf{v}_n);$$

(iii) for any $i = 1, 2, \dots, n$, and any $\mathbf{v}'_i, \mathbf{v}''_i$ we have

$$(2.7) \quad \mathbf{v}_1 \times \mathbf{v}_2 \times \dots (\mathbf{v}'_i + \mathbf{v}''_i) \times \dots \times \mathbf{v}_n = \mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}'_i \times \dots \times \mathbf{v}_n + \\ + \mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}''_i \times \dots \times \mathbf{v}_n;$$

Remark. 2.2 These properties (i)–(iii) denote that this vectorial product is a n -linear mapping, total-antisymmetric on the vectorial space $(R^{n+1})^n$,

(iv) if $\mathbf{v}_i$ and $\mathbf{v}_j$, $i \neq j$, are proportional then the vectorial product is zero;

(v) if the vectors $\mathbf{v}_i$ are differentiable functions of a parameter t , then the vectorial product is differentiable and we have

$$(2.8) \quad \frac{d}{dt} \mathbf{a}(t) = \overset{(1)}{\mathbf{a}}(t) = \sum_{i=1}^n \mathbf{v}_1 \times \mathbf{v}_2 \times \dots \times \mathbf{v}_{i-1} \times \overset{(1)}{\mathbf{v}_i} \times \mathbf{v}_{i+1} \times \dots \times \mathbf{v}_n;$$

By $\overset{(k)}{\mathbf{v}}(t)$ we denote the derivative of order k of the vector $\mathbf{v}(t)$.

(vi) the inner (scalar) product of $\mathbf{a}$ by $\mathbf{u} \in R^{n+1}$ have the form

$$(2.9) \quad \mathbf{a} \mathbf{u} = \sum_{k=1}^{n+1} a_k u_k = \det (\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n);$$

that is this inner product is an extension of the classical *mixt product* in R^2 .

The proof of these properties is evident.

3. Properties of a nonlinear differential system

Let in R^{n+1} be considered the nonlinear differential system

$$(3.1) \quad \overset{(n)}{\mathbf{x}} = \sum_{k=0}^{n-2} \overset{(1)}{F_k}(\overset{(1)}{\mathbf{x}}, \overset{(2)}{\mathbf{x}}, \overset{(n-1)}{\mathbf{x}}, t) \overset{(k)}{\mathbf{x}}, \quad t \in I,$$

with the Cauchy initial dates

$$(3.2) \quad \overset{(j)}{\mathbf{x}}(t_0) = \overset{(j)}{\mathbf{x}}_0, \quad j = 0, 1, 2, \dots, n-1$$

where $\overset{(i)}{\mathbf{x}} = \overset{(i)}{\mathbf{x}}(t) \in R^{n+1}$, $i = 0, 1, 2, \dots, n$, $t_0 \in I$, I is an interval of R and $F_k: (R^{n+1})^n \times I \rightarrow R$.

Theorem 3.1 If the system (3.1), (3.2) has solutions then

1. any solution $\mathbf{x} = \mathbf{x}(t)$ represents in R^{n+1} a plane curve;
2. the plane of the trajectory is determined by Cauchy initial dates (3.2) only, that is the plane is the same for any solution of (3.1), (3.2);
3. the origin $\mathbf{o}$ belongs to this plane.

Proof: Let $\mathbf{x} = \mathbf{x}(t)$ be a solution of (3.1), (3.2) and let $\overset{(1)}{\mathbf{x}}, \overset{(2)}{\mathbf{x}}, \dots, \overset{(n-1)}{\mathbf{x}}$, be the first $(n-1)$ derivatives [1]. By (3.2) we have

$$(3.3) \quad \overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \overset{(n-2)}{\mathbf{x}} \times \overset{(n)}{\mathbf{x}} = \sum_{k=0}^{n-2} F_k \overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \dots \times \overset{(n-2)}{\mathbf{x}} \times \overset{(k)}{\mathbf{x}}.$$

But in the second part of any k two vectors are equal and then

$$(3.4) \quad \overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \overset{(n-2)}{\mathbf{x}} \times \overset{(n)}{\mathbf{x}} = 0.$$

By (2.8) we have

$$(3.5) \quad \frac{d}{dt} (\overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \overset{(n-2)}{\mathbf{x}} \times \overset{(n-1)}{\mathbf{x}}) = \\ = \overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \overset{(n-2)}{\mathbf{x}} \times \overset{(n)}{\mathbf{x}} = 0$$

and then

$$(3.6) \quad \overset{(1)}{\mathbf{x}} \times \overset{(2)}{\mathbf{x}} \times \dots \times \overset{(n-2)}{\mathbf{x}} \times \overset{(n-1)}{\mathbf{x}} = \mathbf{C},$$

where $\mathbf{C} \in R^{n+1}$ is a constant vector.

Because $\mathbf{x} = \mathbf{x}(t)$ is a solution of (3.1), (3.2) we have by

(3.2) that $\mathbf{C} = \overset{(1)}{\mathbf{x}}_0 \times \overset{(2)}{\mathbf{x}}_0 \times \overset{(n-1)}{\mathbf{x}}_0$, that is $\mathbf{C}$ is determined by Cauchy initial dates only.

From (3.6) and the property (vi, § 2) we have

$$(3.7) \quad \mathbf{C} \mathbf{x} = 0$$

or $C_1 x_1(t) = C_2 x_2(t) + \dots + C_{n+1} x_{n+1}(t) = 0$, that is the trajectory corresponding to the solution $\mathbf{x} = \mathbf{x}(t)$ is a plane curve in R^{n+1} and the theorem is proved.

4. Some applications

Let in R^3 be the vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and a function $F: R^3 \times R^3 \times I \rightarrow R$ be considered.

We shall consider the nonlinear equation

$$(4.1) \quad \ddot{\mathbf{r}} = F(\mathbf{r}, \dot{\mathbf{r}}, t) \mathbf{r}, \quad t \in I,$$

which represents the Newton equation for a material point. $F \mathbf{r}$ denotes in mechanics a central force with the centrum $\mathbf{0}$.

Let $\mathbf{r}_0$ and $\dot{\mathbf{r}}_0$ be the Cauchy initial dates which denotes in mechanics the initial position and the initial velocity of the material point.

The equation (3.1) is of (3.1) type and then we have that by central forces (particularly by Newtonian attraction force) we can obtain characteristics for the trajectory *ohne* the effective finding of the solution of (4.1).

The theorem 3.1 proves that the trajectory is in a plane, this plane contains the centrum of the force and that this plane is determined by the initial position and velocity only.

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SOME CONSIDERATIONS CONCERNING THE DECODING OF SYSTEMATIC LINEAR CODES

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Introduction

It is a well-known fact that a d - minimum distance linear code can only detect a number of $d - 1$ errors at the most, and correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ random errors.

It is also known that a linear code is given by its generator matrix $G_{k,n}$, to which the parity-check matrix $H_{n-k,n}$ is associated.

The aim of this paper is to present some considerations regarding the decoding of systematic linear codes for which, as will be shown further down, it is sufficient to merely decode the errors in k codeword positions to which linear independent columns correspond in the generator matrix, while the errors in the other positions and therefore, implicitly, those in the information positions, result from a simple syndrome calculation.

It is known that a d - distance linear code has this peculiar property that any $d - 1$ columns in its parity check matrix $H_{n-k,n}$ are linearly independent.

For a systematic linear code, we know the generator matrix $G_{k,n}$ has, in the assumption that the information positions are the first k positions in the codewords, the canonical form $G_{k,n} = (I_{k,k} \mid A_{k,n-k})$ to which the parity check matrix $H_{n-k,n} = (-A_{k,n-k}^t \mid I_{n-k,n-k})$ corresponds, where t means the transposed of the respective matrix. It is obvious that k columns in the generator matrix $G_{k,n}$, corresponding to the information positions, are linear independent and so, are, in the assumed hypothesis, the last $n - k$ columns in the parity check matrix $H_{n-k,n}$.

For decoding systematic linear codes, the decoding of the errors in information positions is sufficient, and any decoding method must finally be capable of achieving first that. However, it often occurs that errors in those positions cannot be directly decoded, but their decoding results immediately when the errors in other word positions are known.

It will be shown in this article that the knowledge of the errors in any k positions of codewords, to which linearly independent columns (the so-called information sets) correspond in the generator matrix is sufficient for decoding all errors (we are situated in the case of correctable errors). So it may be seen that we do not necessarily impose a direct determination of the errors in the information positions (which in many cases, is rather difficult to do), any decoding algorithm must therefore either find out k linearly independent positions (by misuse of language) which are free of errors (if such positions do exist) or be able to decode

the errors in k linearly independent positions which are more easily decodable.

The material is drawn up in five sections, as follows: the first section contains some considerations regarding the existence of information sets, the second section contains some considerations concerning syndrome calculation, the third section deals with some basic elements of Yau-Liu separable maximum-distance code decoding algorithm, the fourth section shows the feasibility of extending this algorithm to some systematic linear codes, and the fifth section brings up the conclusions to be drawn from the whole material.

Reference is made now and then along this work, to systematic cyclic codes.

I. Sets of information and sets of k linearly independent columns in the generator matrix

There is a close relationship between the generator matrix $G_{k,n}$ of a linear code and the parity-check matrix $H_{n-k,n}$, namely it is known that:

$$G_{k,n} \cdot H_{n-k,n}^t = 0$$

For the systematic linear codes where matrices $G_{k,n}$ and $H_{n-k,n}$ have the form:

$$G_{k,n} = (I_{k,k} \mid A_{k,n-k}) \quad (1)$$

and

$$H_{n-k,n} = (-A_{k,n-k}^t \mid I_{n-k,n-k}) \quad (2)$$

it can be observed that the following proposition can be set forth:

Proposition 1

If k columns in the generator matrix are linearly independent, then the remnant $n - k$ columns in the parity-check matrix are also linearly independent and the inverse is also true.

Proof:

This results from form (2) of the parity-check matrix directly obtained from (1). Q.E.D.

Definition 1

A set K of information is a set of code positions $\alpha_1, \alpha_2, \dots, \alpha_k$ such that a single code can be constructed for every one of the q^k various assignments of the component of the vector of those positions.
(See W.C. Gore [3])

It is obvious that those columns of $G_{k,n}$ which correspond to an information set are linearly independent and each linearly independent set of k columns in the generator matrix correspond to an information set.

Proposition 2

Assume a given linear code $V_{n,k}$ of minimum distance d .
For each information set, let us mark it

$$K^{(0)} = \{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_k\},$$

there are at least $d - 1$ different information set K which have $k - 1$ positions in common, $\alpha_1, \alpha_2, \dots, \alpha_{k-1}$ while the k^{th} position differs (See W.C. Gore [3])

Proposition 3

In any linear cyclic code, any set of k consecutive codeword positions (where also the i positions at the tail of the word with the $k - i$ positions from the beginning of the word are consecutive, $i = 1, 2, \dots, k - 1$) is an information set (connected information set).

Proof:

If there were k consecutive columns in the generator matrix which were not linearly independent then nonzero code words would be obtained with zeros in succession on k positions, a thing which is impossible in a cyclic codes, as these k positions may by cyclically permutations, be moved against the k information positions and nonzero codewords would be obtained, with all the information symbols being null, contradiction (It is known that the only word with null information positions is the null codeword)

Q.E.D.

Corollary 1

Any cyclic code detects any burst of errors of a length equal to $n - k$ (consequently, excepting optimal codes, a length longer than the $d - 1$ value which can be ensured by any code).

Proof:

Any burst of errors of a length $n - k$ may be considered as a non-zero word with k consecutive null positions, and this cannot be a code word as, in a cyclic code, according to Proposition 3, any set of k positions in sequence is an information set, and the only codeword which can contain zeros in this information set is the null codeword

Remark:

We have shown this proof as an alternative to other, much more complicated demonstrations which can be found in the specialized literature.

Corollary 2:

There exist no line having more than $k - 1$ zero positions in sequence in the generator matrix of any cyclic code.

Proof:

Every line in the generator matrix is a codeword in the cyclic code generated by that matrix.

Q.E.D.

Proposition 3 can also be stated as the equivalent:

Proposition 4

The set of cyclic codes is a subset of linear codes in which all sets of k consecutive positions are information sets.

With optimal linear codes we have an extreme case in which $(\forall)$ k codeword positions are information positions this is why we recall that:

Definition 2

A linear code $V_{n,k}$ is named an optimal linear code (maximum-distance linear code) if $d = n - k + 1$

Proposition 5

A linear code $V_{k,n}$ over $GF(q)$ is an optimal linear code if, and only if, any k codeword positions (not necessarily consecutive) can be considered as information positions (information set), while the remnant $n - k$ positions can be considered as control positions.

Proof:

For the implication:

Optimal code $\Rightarrow (\forall)$ k positions may be considered as being information positions see Singleton [5].

For the systematic optimal codes, an alternative to Singleton's proof of this implication is the following:

Assume $d = n - k + 1$. In the $H_{n-k,n}$ matrix, any $d - 1 = n - k$ columns are linearly independent. Since, according to Proposition 1, to $(\forall)$ $n - k$ linearly independent columns in $H_{n-k,n}$ correspond the other k linearly independent columns in $G_{k,n}$, it results that $(\forall)$ k columns from the generator matrix $G_{k,n}$ are linearly independent, and therefore $(\forall)$ k codeword positions may be considered as information positions.

For the other implication see Yau and Liu [7].

Remark:

It is to be observed that in optimal linear code there is no codeword having more than $k-1$ zeros.

II. Syndrome, associated syndrome with k positions

Given a linear code $V_{n,k}$ with a parity-check matrix $H_{n-k,n}$, the definition is known of a syndrome $\bar{s}$ of a received word $\bar{r}$, as being the $(n - k)$ - dimensional vector $\bar{s}$ given by the relation:

$$\bar{s} = H_{n-k,n} \cdot \bar{r} = H_{n-k,n} \cdot (\bar{c} + \bar{e}) = H_{n-k,n} \cdot \bar{e} \quad (4)$$

where: $\bar{c}$ is the transmitted codeword vector.

and $\bar{e}$ is the error vector (error pattern).

For the cyclic codes, the definition of the $s(x)$ polynomial syndrome is known, which is given by:

$$s(x) = R_g[r(x)] = R_g[e(x)] = e(x) + c(x) - \tilde{c}(x) \quad (5)$$

where: $R_g[r(x)]$ means the remainder when the received word $r(x)$ is divided by cyclic code generator $g(x)$, $\tilde{c}(x)$ is a codeword having, as information symbols, the symbols in the information positions of the received word $r(x)$, and $c(x)$ is the codeword polynomial transmitted with highest order symbols first, i.e. c_{n-1} is the first transmitted symbol, c_{n-2} is the second, etc.

Proposition 6

The definitions given by relations (4) and (5) are equivalent for systematic cyclic codes, while for systematic linear codes, using the vectorial notation, the following definition may be used:

$$\hat{s} = \bar{e} + \bar{c} - \tilde{r}$$

where $\bar{s}$ is obtained from $\hat{s}$ by removing the k zeros at the front

Proof:

Indeed, it is known that, using the definition given by the first part of relation (5):

$$s(x) = R_g[r(x)] = R_g[e(x)]$$

we obtain:

$$s(x) = e(x) + c(x) - \tilde{c}(x)$$

Using the vectorial notation and the definition given by relation (4), we obtain, using also relation (2):

$$\bar{s} = (\tilde{H}_{n-k, k} \mid I_{n-k, n-k}) (\bar{r}_i + \bar{r}_p) =$$

where $\bar{r}_i$ of a length n has zeros on the last $n - k$ positions, and $\bar{r}_p$ of the same length has zeros on the first k position and $\bar{r} = \bar{r}_i + \bar{r}_p$

$$= \tilde{H}_{n-k, k} \cdot \bar{r}'_i + \bar{r}'_p =$$

where $\bar{r}'_i$ and $\bar{r}'_p$ are $\bar{r}_i$ and $\bar{r}_p$ respectively with the last $n - k$, the first k respectively, zeros removed.

$$= \tilde{H}_{n-k, k} \cdot \bar{r}'_i + \bar{r}'_p + \bar{r}'_p - \bar{r}'_p =$$

$$= H_{n-k, n} \cdot (\bar{r}_i + \tilde{r}_p) + \bar{r}'_p - \tilde{r}'_p = \bar{r}'_p - \tilde{r}'_p \quad (6)$$

because $\tilde{c} = \bar{r}_i + \tilde{r}_p = \text{codeword}$

($\tilde{r}_p$ was taken as such)

If we extend $\bar{s}$ by k zeros in front, noting it as $\hat{s}$, we obtain:

$$\begin{aligned} \hat{s} &= \bar{r}_p - \tilde{r}_p + \bar{r}_i - \bar{r}_i = \bar{r}_i + \bar{r}_p - (\bar{r}_i + \tilde{r}_p) = \bar{r} - \tilde{c} = \\ &= \bar{c} + \bar{e} - \tilde{c} = \bar{e} + (\bar{c} - \tilde{c}) \end{aligned}$$

Therefore :

$$\hat{s} = \bar{e} + \bar{c} - \bar{c} \quad (7)$$

that is, the same formula as before with the definition (5)

It may be seen that, in obtaining formula (7) we did not make use of the fact that the code is cyclic but just of the fact that it is systematic linear code.

Q.E.D.

It is to be remarked that :

$$w[\hat{s}] = w[\bar{s}] \quad (8)$$

where $w[\]$ means the Hamming weight of the respective vector. Relation (8) will be useful further on.

The following proposition is known :

Proposition 7

For a systematic cyclic code correcting t errors (consequently of a minimum distance $d \geq 2t + 1$), the errors only appeared in the parity positions (redundant positions) if and only if the Hamming weight of the $s(x)$ syndrome complies the relation :

$$w[s(x)] \leq t \quad (9)$$

(See Szwaja [6])

We shall show that relation (9) occurs not only for systematic cyclic codes but for any systematic linear code.

Indeed, in [7], Yau and Liu submit, without demonstrating it, the following proposition :

Proposition 8

In any systematic linear code $V_{n,k}$, the weight of the syndrome of a received word in which t errors have occurred, $2t + 1 \leq d$, is lower or equal to t if, and only if, the errors have occurred in the control positions (redundant positions).

Proof :

Taking a systematic linear code where we assume, without losing anything from the generality, that the first k positions are information positions, consequently with :

$$H_{n-k,n} = (\tilde{H}_{n-k,n} \mid I_{n-k,n-k})$$

We know that the syndrome of a received word $\bar{r}$ is, by definition

$$\begin{aligned} \bar{s} &= H_{n-k,n} \cdot \bar{r} = H_{n-k,n} \cdot (\bar{c} + \bar{e}) = \\ &= H_{n-k,n} \cdot (\bar{e}_i + \bar{e}_p) = H_{n-k,n} \cdot \bar{e}_i + H_{n-k,n} \cdot \bar{e}_p = \\ &= H_{n-k,n} \cdot \bar{e}_i + \bar{e}'_p \end{aligned}$$

where : $\bar{e}_i$ is a vector of a length n with zeros on the last $n - k$ positions ;
 $\bar{e}_p$ is a vector of a length n with zeros on the first k positions, and
 $\bar{e}_p'$ is the $\bar{e}_p$ vector in which the first k zero positions are removed,
 therefore, a $(n - k) -$ dimensional vector.

Let us consider $w[\bar{e}] \leq t$

Consequently :

$$w[\bar{e}_i + \bar{e}_p] = w[\bar{e}_i] + w[\bar{e}_p]$$

Taking $\bar{e}_i = 0$, that is $w[\bar{e}_i] = 0$ and $\bar{e}_p \neq 0$
 (therefore errors only in control positions), as

$H_{n-k, n} \cdot \bar{e}_i = 0$, it results that

$$w[\bar{s}] = w[\bar{e}_p'] \leq t$$

Conversely :

Given $w[\bar{s}] \leq t$. Show that $\bar{e}_i = 0$

We have seen that, for systematic linear codes :

$$\hat{s} = \bar{e} + \bar{c} - \tilde{c}$$

and $w[\hat{s}] = w[\bar{s}]$

Assume that $\bar{e}_p = 0$ („per absurdum”)

Therefore,

$$w[\hat{s} - \bar{e}] = w[\hat{s} - \bar{e}_i] = w[\bar{c} - \tilde{c}] \geq d \geq 2t + 1$$

But $w[\hat{s} - \bar{e}_i] = w[\hat{s}] + w[\bar{e}_i]$

Therefore,

$$w[\hat{s}] + w[\bar{e}_i] \geq 2t + 1$$

Hence

$$w[\hat{s}] = w[\bar{s}] \geq 2t + 1 - t = t + 1 > t$$

which is a contradiction, as we have assumed that

$$w[\bar{s}] \leq t.$$

Consequently $\bar{e}_p \neq 0$ and $\bar{e}_i = 0$ or $\bar{e}_i \neq 0$

Let us show that $\bar{e}_i = 0$

Given $b = w[\bar{e}_p]$ and $\bar{e}_i \neq 0$, hence

$0 < w[\bar{e}_i] \leq t - b$, we have :

$$w[\bar{e}_i - \hat{s}] = w[\bar{e}_i] + w[\hat{s}] = w[\tilde{c} - \bar{c} - \bar{e}_p] \geq 2t + 1 - b$$

because $\bar{e} - \hat{s} = \tilde{c} - \bar{c}$

hence $\bar{e}_i + \bar{e}_p - \hat{s} = \tilde{c} - \bar{c}$

hence $\bar{e}_i - \hat{s} = \tilde{c} - \bar{c} - \bar{e}_p$

Consequently, for $0 < w[\bar{e}_i] \leq t - b$, we have :

$$w[\bar{s}] = w[\hat{s}] \geq 2t + 1 - b - (t - b) = t + 1 > t$$

which is a contradiction, hence $\bar{e}_i = 0$ and $\bar{e} = \bar{e}_p$

Q.E.D.

Stephen S. Yau and Yu-Cheng Liu, studying the decoding of maximum-distance separable linear codes, introduced the concept of syndrome associated with any given k codeword positions.

Thus :

Definition 3

A syndrome associated with any given k codeword positions $\alpha_1, \alpha_2, \dots, \alpha_k$, is the expression :

$$\bar{s}_\alpha = \begin{bmatrix} r'_{\alpha_{k+1}} \\ \vdots \\ r'_{\alpha_n} \end{bmatrix} - \begin{bmatrix} r_{\alpha_{k+1}} \\ \vdots \\ r_{\alpha_n} \end{bmatrix} = (G_\alpha^t \parallel I_{k,k}). \begin{bmatrix} r_{\alpha_1} \\ \vdots \\ r_{\alpha_k} \\ r_{\alpha_{k+1}} \\ \vdots \\ r_{\alpha_n} \end{bmatrix} \quad (11)$$

where :

$$\begin{bmatrix} r'_{\alpha_{k+1}} \\ \vdots \\ r'_{\alpha_n} \end{bmatrix} = G_\alpha^t \cdot \begin{bmatrix} r_{\alpha_1} \\ \vdots \\ r_{\alpha_k} \end{bmatrix} \quad (12)$$

G_α = the generator matrix of the dimension $k \times (n - k)$ corresponding to the information positions $\alpha_1, \alpha_2, \dots, \alpha_k$, that is the matrix for which :

$$(c_{\alpha_{k+1}}, \dots, c_{\alpha_n}) = (c_{\alpha_1}, \dots, c_{\alpha_k}) \cdot G_\alpha \quad (13)$$

$I_{k,k}$ is, as before, the identity matrix $k \times k$

Remark : If $\alpha_1, \alpha_2, \dots, \alpha_k$ form an information set, it will be observed that formula (11) is the same as formula (5) with the exception of the sign.

Besides, we shall see that the G_α matrix, hence also $\bar{s}_\alpha$, can only be calculated for information sets.

Yau and Liu have also indicated the way in which G_α should be calculated, but only for separable optimal codes, where any submatrix $(n - k) \times (n - k)$ of the parity-check matrix $H_{n-k, n}$ is inversable. Thus, in [7] Yau and Liu give the following proposition:

Proposition 9

For maximum-distance separable linear codes the generator matrix G_α corresponding to any given k positions $\alpha_1, \alpha_2, \dots, \alpha_k$ is directly calculated from the parity-check matrix $H_{n-k, n}$, namely with the relation:

$$G_\alpha = - \begin{bmatrix} \bar{h}_{\alpha_1} \\ \vdots \\ \bar{h}_{\alpha_k} \end{bmatrix} \cdot \begin{bmatrix} \bar{h}_{\alpha_{k+1}} \\ \vdots \\ \bar{h}_{\alpha_n} \end{bmatrix}^{-1} \quad (41)$$

where $\bar{h}$ are lines of the $H_{n-k, n}^t$ matrix, therefore vectors of length $n - k$

III. Some basic elements of Yau-Liu's algorithm

Yau and Liu have given a decoding algorithm for maximum-distance separable linear codes, which consists in searching k positions free of errors and obtaining the errors from the other positions using a very simple calculation.

The basic elements in applying this algorithm are the following two propositions:

Proposition 10

In the case of maximum-distance separable linear code, any k positions $\alpha_1, \alpha_2, \dots, \alpha_k$ of a received word $\bar{r}$ are free of errors if, and only if, the syndrome $\bar{s}_\alpha$ associated with those positions satisfies the condition:

$$w[\bar{s}_\alpha] \leq t \text{ where } t \leq \left\lfloor \frac{n - k}{2} \right\rfloor$$

Proof: For systematic optimal linear codes it results directly from Proposition 8 as, in the case of systematic optimal linear codes, $(\nabla) k$ positions may be considered as information positions and the other as parity positions. For the case of maximum-distance separable linear codes see Yau and Liu [7].

Proposition 11

In a maximum-distance separable linear code the error vector $\bar{e}$, in which positions $\alpha_1, \alpha_2, \dots, \alpha_k$ are free of errors, is given by $\bar{s}_\alpha$, namely:

$$\bar{e}'_\alpha = -\bar{s}_\alpha \quad (15)$$

where $\bar{e}'_\alpha$ is the $\bar{e}$ error vector from which the zero elements in positions $\alpha_1, \alpha_2, \dots, \alpha_k$ have been removed.

(See Yau and Liu [7])

The Yau-Liu algorithm presented by its authors for maximum-distance separable linear codes only, reduces itself to the search of an information set free of errors using Proposition 10 as a basis when, once this set has been found, the error vector is immediately obtained through Proposition 11. The first associated syndrome $\bar{s}_\alpha$ with the weight $\leq \left\lceil \frac{d-1}{2} \right\rceil$ gives the k positions free of errors, the other positions being immediately decoded, being considered as parity positions.

We do not lay before our readers the simplifications given by Yau and Liu in order to avoid calculating a great number of associated syndromes $\bar{s}_\alpha$ until an information set free of errors is found, as those simplifications can be found in their work [7].

Let us see what simplifications would appear in applying the Yau-Liu algorithm to the decoding of systematic optimal cyclic codes (e.g. the Reed-Solomon codes)

For systematic optimal cyclic codes the decoding algorithm would be as follows :

Step 1

Test wheather there are k consecutive positions free of errors, by cyclic shifts of the received word, calculating the syndrome polynomial each time, using the recursive formula :

$$\tilde{s}(x) = R_q [xs(x)] \quad (16)$$

where : $\tilde{s}(x)$ is the syndrome polynomial when the word has been cyclically shifted to the left by one position as against the previous situation when the syndrome polynomial was $s(x)$.

This thing is done until a syndrome polynomial $s(x)$ is found having the weight $\leq \left\lceil \frac{d-1}{2} \right\rceil$. If no syndrome is found, through the n successive cyclic shifts, with the weight $\leq \left\lceil \frac{d-1}{2} \right\rceil$, then :

Step 2

Yau-Liu algorithm is applied for searching any k positions free of errors which obviously form an information set (due to the code's optimality).

As we shall further see, there are always k positions free of errors (in the situation when the number of errors $\leq \left\lceil \frac{d-1}{2} \right\rceil$) for any linear, not necessarily optimal, code.

IV. An extension of Yau-Liu's algorithm to other linear, not necessarily optimal, codes

A set of codes, not necessarily optimal, capable of being decoded with Yau-Liu's algorithm are the systematic BCH codes.

In this respect we wish to recall that Forney Jr. [1] showed that any BCH code over $GF(q)$ of a length $q^m - 1$ is a subcode subfield of a Reed-Solomon code over $GF(q^m)$. Therefore the parity-check matrix of the respective Reed-Solomon code, which has elements from $GF(q^m)$ (the generator polynomial has its roots from $GF(q^m)$) may also be considered as being a parity-check matrix for the BCH code, subcode subfield over $GF(q)$ of the Reed-Solomon code.

We therefore may enunciate the following proposition :

Proposition 12

Any systematic, BCH code can be decoded, in with Yau-Liu's algorithm.

As far as systematic codes are concerned, we may enunciate the following proposition :

Proposition 13

Any systematic linear code can only be decoded with the aid of syndrome calculation if for each received word an information set free of errors can be found or if, through some algorithm, positions can be decoded in such a manner as to form an information set.

Proof :

Finding an information set implies the use of Yau-Liu's algorithm which, as we have seen, is an algorithm for searching k positions free of errors.

If we demonstrate that G_α can be computed for any information set, then we have finished.

But we know that, if $\alpha_1, \alpha_2, \dots, \alpha_k$ form an information set, the corresponding columns in the generator matrix $G_{k;n}$ are linearly independent and, through Proposition 1, also the rest of $n - k$ columns in the parity-check matrix $H_{n-k,n}$ are linearly independent, therefore the expression :

$$\begin{bmatrix} \bar{h}_{\alpha_{k+1}} \\ \vdots \\ \bar{h}_{\alpha_n} \end{bmatrix}^{-1}$$

has a sens and hence G_α can be computed by using relation (14).

Q.E.D.

We may therefore say that everything consists in :

a) either being able to found, for each received word, an information set free of errors (for the cyclic codes it is sufficient just to find k consecutive positions free of errors if such positions exist).

b) or to decode any k positions which, however must form an information set.

Let us study both these situations in succession.

The first situation — Find out when an information set free of errors can be found

In addition to optimal codes, there are many other codes for which an information set free of errors can be found :

First of all, we must see how many error free positions can exist in a code in the situation when correctible errors occur.

Lema 1

The number of codeword positions free of errors in any received word is $\left\lfloor \frac{n+k}{2} \right\rfloor > k$ as a minimum.

Proof :

We know that the correctible errors have a weight of $\left\lfloor \frac{d-1}{2} \right\rfloor$ at a maximum. We also know that $d \leq n - k + 1$ for any linear code. Therefore the number of positions in error is $\left\lfloor \frac{d-1}{2} \right\rfloor \leq \left\lfloor \frac{n-k+1-1}{2} \right\rfloor = \left\lfloor \frac{n-k}{2} \right\rfloor$ at a maximum. Therefore the maximum number of positions free of errors in a codeword is :

$$n - \left\lfloor \frac{n-k}{2} \right\rfloor = \left\lfloor \frac{n+k}{2} \right\rfloor > k$$

Q.E.D.

Now we can state the following proposition :

Proposition 14

If, in a systematic linear code, among $(\forall) \left\lfloor \frac{n+k}{2} \right\rfloor$ positions, there exists at least one information set, then the code is decodable through Yau-Liu's error-free information set searching algorithm.

Proof :

In every code, the at least $\left\lfloor \frac{n+k}{2} \right\rfloor$ positions free of errors generally differ from one received word to another. Consequently, if in $(\forall) \left\lfloor \frac{n+k}{2} \right\rfloor$ positions, hence also in $(\forall) \left\lfloor \frac{n+k}{2} \right\rfloor$ positions free of errors, an information set can be found, then, obviously, Yau-Liu's algorithm for searching an information set free of errors can be applied.

Q.E.D.

Examples of codes complying with Proposition 14 are presented in the following two propositions :

Proposition 15

Assume a systematic linear code $V_{n,k}$.

If $k \leq \left\lfloor \frac{n-k}{2} \right\rfloor$, and if among $(\forall) \left\lfloor \frac{n-k}{2} \right\rfloor$ columns in the generator matrix $G_{k,n}$, corresponding to control positions, there exist k linearly independent columns, then in $(\forall) \left\lfloor \frac{n+k}{2} \right\rfloor$ positions, hence also in the at least $\left\lfloor \frac{n+k}{2} \right\rfloor$ positions free of errors in every received word, there is at least an information set, so that Yau-Liu's algorithm for searching an information set free of errors can be applied.

Proof:

We have $\left\lfloor \frac{n+k}{2} \right\rfloor = \left\lfloor k + \frac{n-k}{2} \right\rfloor$

As $k \leq \left\lfloor \frac{n-k}{2} \right\rfloor \Rightarrow$ among the $\left\lfloor \frac{n+k}{2} \right\rfloor$ columns there are either the first k linearly independent columns or, at least $\left\lfloor \frac{n-k}{2} \right\rfloor$ columns from the last $n-k$, which contain k linearly independent columns in conformity with the assumption.

Q.E.D.

Proposition 16

If $k \leq \left\lfloor \frac{n-k}{2} \right\rfloor$, and if $(\forall) k$ columns in the control part of $G_{k,n}$ are linearly independent (for example in the case when $n-k$ columns from $G_{k,n}$, taken in combinations of k columns each, form Vandermonde — type determinants — optimality in the control part of $G_{k,n}$), then an information set can be found among the $(\forall) \left\lfloor \frac{n+k}{2} \right\rfloor$ codeword positions, hence the Yau-Liu algorithm for searching an error free information set can be applied.

Proof.

Either the first k information positions or k from the $n-k$ control positions are among the $\left\lfloor \frac{n+k}{2} \right\rfloor = \left\lfloor k + \frac{n-k}{2} \right\rfloor$ positions because

$$k \leq \left\lfloor \frac{n-k}{2} \right\rfloor$$

Q.E.D.

The second situation—when an error-free information set each cannot be found

In this case such algorithm must be searched or used as can decode the errors in that set of information which presents some facilities in decoding, and then, using the above (11) and (15) relations, given by Yau and Liu, the errors in all the other positions, hence also those in the information positions are immediately obtained.

In the case of systematic cyclic codes, an algorithm only decoding any given position suffices, because the errors in k positions can be decoded by cyclical shifts and, as $(\forall) k$ consecutive positions in a cyclic code form an information set, the errors in all the other positions can be decoded by simple calculation, applying relations (11) and (15).

Nevertheless, for the decoding of systematic cyclic codes, any general decoding algorithm must contain the following steps :

Step 1 :

Testing whether there are k consecutive positions free of errors. If they exist, we have finished.

Step 2

Applying, if k consecutive positions free of errors do not exist, one among the algorithms used for decoding one single position when, by circular shift, the errors in k consecutive positions can be decoded

Step 3

Applying relations (11) and (15) when the errors in all positions, hence also in the information positions, are obtained.

V. Conclusions

Using some elements in Yau-Liu's algorithm, the conclusion is reached that, for systematic linear code decoding any algorithm must, either be capable of detecting an information set free of errors in each of the received words (if any such information set exist) or be capable of decoding an information set displaying some decoding facilities.

Once an error-free information set is known, all the other positions in the received codeword are immediately decoded by rather simple calculation.

For the systematic cyclic codes, it suffices to obtain k consecutive positions (which, as we have seen, always form an information set) free of errors for decoding the errors from all the other positions hence, also from the information positions.

* * *

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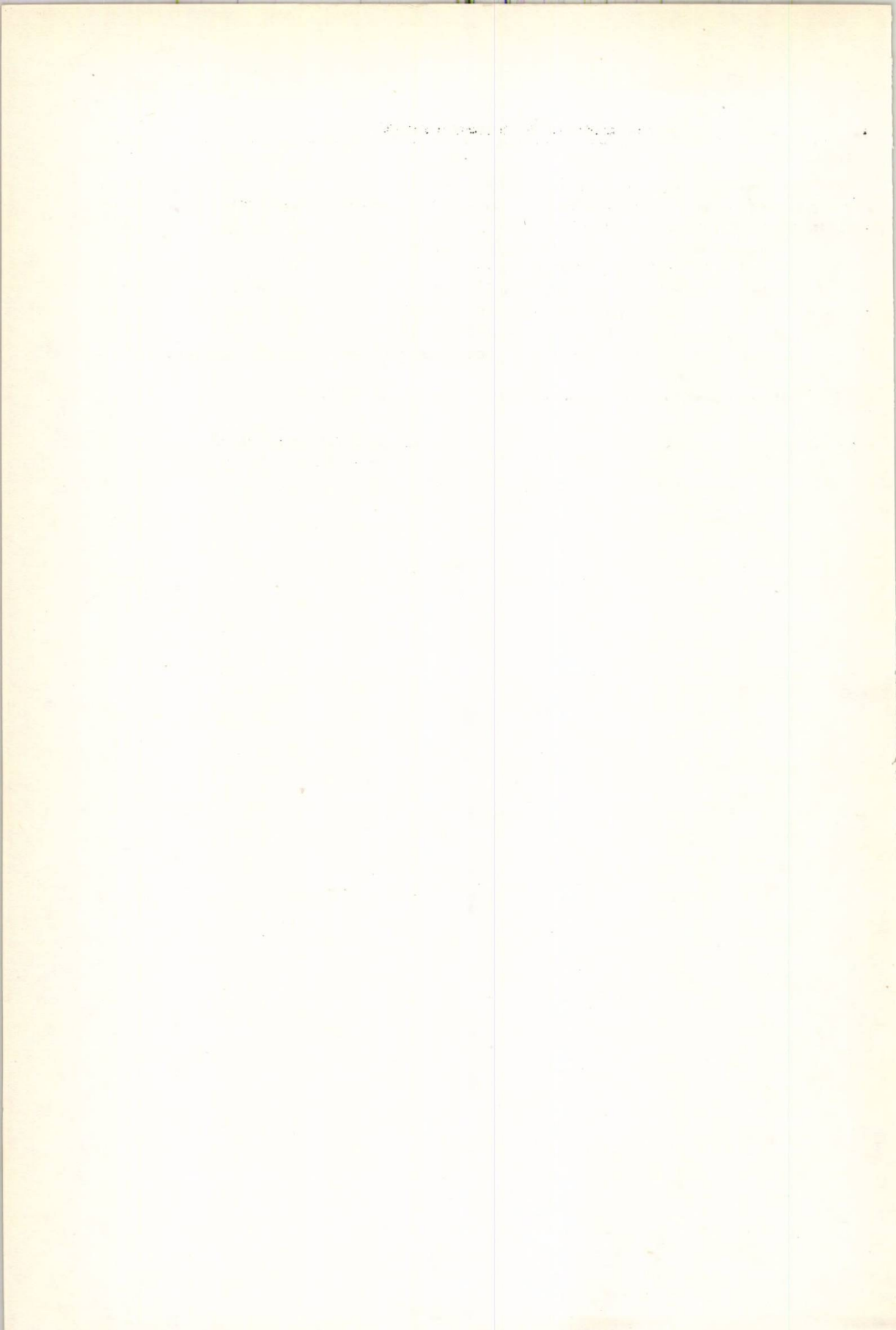
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ON THE MOTION OF PERFECT FLUIDS IN THE PRESENCE OF AN INHOMOGENEOUS POROUS BODY

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1. The steady motion of fluids in the presence of an isotropic homogeneous or inhomogeneous porous body has been considered in several papers beginning from 1954 [1], and a survey of studied problems can be found in [2]. In the case of perfect fluids the inhomogeneity was of the second type (body homogeneous by parts) or of the type 0 (body having a cavity), but in the case of viscous fluids in slow motion, besides the mentioned inhomogeneities, it has been considered two particular cases of inhomogeneity of the third type, namely

$$k^{1/2} = A + B \ln r, \quad (1)$$

in the plane case, and

$$k^{1/2} = A + Br^{-1} \quad (1')$$

in the three-dimensional case, where k is the filtration coefficient, r is the distance from the origin of the frame of reference, A and B are real constants such that $A + B \ln r$ or $A + Br^{-1}$ are positive defined in the porous body. The case of a homogeneous sphere when the fluid is viscoplastic has been considered in [3].

Here we deal with the steady irrotational motion of a perfect incompressible fluid in the presence of an isotropic inhomogeneous body of the type III, neglecting the influence of the gravity. Therefore, denoting by D_1 the exterior of the body and by D_2 the domain occupied by the body, by φ the velocity potential in D_1 and by p_2 the pressure in D_2 , the first function is harmonic,

$$\Delta \varphi = 0, \quad (2)$$

but p_2 satisfies the equation

$$\operatorname{div}(k \operatorname{grad} p_2) = 0. \quad (3)$$

We shall approximate k in two ways, assuming that

$$\Delta k^{1/2} + a k^{1/2} = 0 \quad (4)$$

or

$$\Delta k^{1/2} = 0. \quad (4')$$

a being a positive constant, i.e. the body is „Helmholtz inhomogeneous” or „harmonically inhomogeneous”. Then, it is known that if we introduce a function F defined by

$$F = p_2 k^{1/2}, \quad (5)$$

then F satisfies the same equation (4) or (4') ($\Delta F + aF = 0$ or $\Delta F = 0$).

A general method of solving such problems is given in § 2, but an application is briefly considered in § 3.

2. The motion in D_1 is described by the velocity $\vec{v}_1$ given by

$$\vec{v}_1 = \text{grad } \varphi \quad (6)$$

and the pressure p_1 , therefore, with the above mentioned assumptions,

$$p_1 = C - \frac{\rho}{2} (\text{grad } \varphi)^2, \quad (7)$$

where C is a constant. All the singularities of motion are in D_1 .

In D_2 the motion is considered to be linear, and denoting by γ the specific weight of the fluid,

$$\vec{v}_2 = -\gamma^{-1} k \text{ grad } p_2. \quad (8)$$

Having in view (5), we can write

$$\vec{v}_2 = \gamma^{-1} (F \text{ grad } k^{1/2} - k^{1/2} \text{ grad } F). \quad (9)$$

On the frontier S between D_1 and D_2 we must satisfy the known matching conditions, the continuity of pressures and the continuity of normal velocities, then, with (5), (7) and (9),

$$C - \frac{\rho}{2} (\text{grad } \varphi)^2 = F k^{-1/2}, \quad (10)$$

$$\gamma^{-1} \left(F \frac{\partial k^{1/2}}{\partial n} - k^{1/2} \frac{\partial F}{\partial n} \right) = \frac{\partial \varphi}{\partial n}. \quad (11)$$

In order to solve this non-linear matching problem, we proceed as follows. The motion in D_2 is in general very slow, and therefore the perturbation of the motion in D_1 , with respect to the motion when S is the frontier of an impervious body, is very small, such that we must consider k as a small parametr. Usually it is more convenient to consider

a dimensionless parameter to be small, and, as in previous papers (see, for instance, [2]), denoting by $\tilde{k}$ a characteristic value of the filtration coefficient, by $\tilde{V}$ a characteristic velocity in D_1 , by L a characteristic length of the body and by g the acceleration of gravity, we can take

$$K = \tilde{k} \tilde{V} / (g L) \quad (12)$$

to be a small parameter.

If k_* is the maximum value of k , we write

$$k = k_* k_0, \quad (13)$$

where k_0 is a known function, such that $0 < k_0 \leq 1$. Then we look for the velocity potential in the form

$$\varphi = \varphi_0 + k_* \varphi_1 + k_*^2 \varphi_2 + \dots \quad (14)$$

and for the function F in the form

$$F = k_*^{1/2} (F_0 + k_* F_1 + k_*^2 F_2 + \dots). \quad (15)$$

With (13)–(15) in (10) and (11) we get the following equations on S :

$$\partial \varphi_0 / \partial n = 0, \quad (16)$$

$$C - \frac{\rho}{2} (\text{grad } \varphi_0)^2 = F_0 k_0^{-1/2}, \quad (17)$$

$$\gamma \frac{\partial \varphi_1}{\partial n} = F_0 \frac{\partial k_0^{1/2}}{\partial n} - k_0^{1/2} \frac{\partial F_0}{\partial n}, \quad (18)$$

$$- \rho \text{grad } \varphi_0 \cdot \text{grad } \varphi_1 = F_1 k_0^{-1/2}, \quad (19)$$

$$\gamma \frac{\partial \varphi_2}{\partial n} = F_1 \frac{\partial k_0^{1/2}}{\partial n} - k_0^{1/2} \frac{\partial F_1}{\partial n}, \quad (20)$$

$$- \rho [(\text{grad } \varphi_1)^2 + 2 \text{grad } \varphi_0 \cdot \text{grad } \varphi_2] = 2F_2 k_0^{-1/2}, \quad (21)$$

and in general

$$\gamma \frac{\partial \varphi_j}{\partial n} = F_{j-1} \frac{\partial k_0^{1/2}}{\partial n} - k_0^{1/2} \frac{\partial F_{j-1}}{\partial n}, \quad (22)$$

$$G(\text{grad } \varphi_0, \text{grad } \varphi_1, \dots, \text{grad } \varphi_j) = F_j k_0^{-1/2}. \quad (23)$$

Therefore first we have to solve an exterior Neumann problem, as pointed out by (16), when we know the singularities in D_1 . Then, by (17), we have to determine a harmonic function F_0 in D_2 when we know its values on the frontier of this domain. The next step is an exterior Neumann problem for a regular harmonic function, followed by an interior boundary value problem, a Dirichlet problem as shown by (19), and so on.

3. As a simple example, let us take the case of a uniform motion at great distances, therefore

$$C = p_0 + \frac{\rho}{2} V_0^2, \quad (24)$$

p_0 and V_0 being the pressure and, respectively, the speed in the point at infinity.

The domain D_2 is the sphere $r \leq R$,

$$k = k_*(b + cr \cos \theta)^2, \quad (25)$$

therefore we have a harmonically inhomogeneous body, ($b + cR = 1$, $b - cR > 0$), but

$$\lim_{r \rightarrow \infty} \vec{v}_1 = V_0 \vec{i}, \quad (26)$$

with the usual notations (θ is the angle between Ox and $\vec{r}$). In the meridian plane,

$$\varphi_0(r, \theta) = V_0 \left(r + \frac{R^3}{2r^2} \right) \cos \theta, \quad (27)$$

and, having in view the axial symmetry, the general expressions of φ , and F , are

$$\varphi = d_0 + \sum d_n P_n (\cos \theta) r^{-(n+1)}$$

$$F = s_0 + \sum s_n P_n (\cos \theta) r^n,$$

where P_n are the Legendre polynomials, but d_i and s_i ($i = 0, 1, 2, \dots$) certain constants which are to be determined by the matching conditions. From (17) we can write

$$F_0(R, \theta) = (b + cR \cos \theta) \left(p_0 - \frac{9}{8} \rho V_0^2 \sin^2 \theta \right),$$

and using known relations (e.g. [4]), we have

$$F_0(R, \theta) = b \left[p_0 - \frac{3}{4} \rho V_0^2 P_2 + p_0 c R P_1 \right] - \frac{9 \rho c R V_0^2}{20} (P_1 - P_3),$$

therefore

$$\begin{aligned} F_0(r, \theta) = & b \left[p_0 - \frac{3}{4} \rho V_0^2 + \frac{3 \rho V_0^2 r^2}{4 R^2} P_2(\cos \theta) \right] + \\ & + c R \left[p_0 \frac{r}{R} P_1(\cos \theta) - \frac{9 r}{20 R} \rho V_0^2 \left(P_1(\cos \theta) - \frac{r^2}{R^2} P_3(\cos \theta) \right) \right]. \end{aligned} \quad (28)$$

As

$$\begin{aligned} \frac{\partial F_0}{\partial r} \Big|_{r=R} = & - \frac{3 \rho b V_0^2}{4 R} + c \left(p_0 - \frac{99 \rho V_0^2}{40} \right) \cos \theta + \frac{9 \rho b V_0^2}{4 R} \cos^2 \theta + \\ & + \frac{27 \rho c V_0^2}{8} \cos^3 \theta, \end{aligned}$$

using (18), we find $\partial \varphi_1 / \partial r$ for $r = R$, and at the end we have

$$\begin{aligned} \gamma \varphi_1(r, \theta) / (\rho V_0^2) = & - \frac{7 b c R^3}{20 r^2} P_1(\cos \theta) + \left(b^2 + \frac{3 c^2 R^2}{14} \right) \frac{R^3}{2 r^3} P_2(\cos \theta) + \\ & + \frac{53 b c R^5}{160 r^4} P_3(\cos \theta) + \frac{18 c^2 R^7}{175 r^2} P_4(\cos \theta). \end{aligned} \quad (29)$$

It is observed that for the motion in D_1 there is no source in D_2 . If we wish to find out $F_1(r, \theta)$, we have to use (19), then φ_2 is determined by (20), and so on.

It is interesting to compare this result with the case of a uniform stream in the presence of a homogeneous sphere, when it has been found, in terms of the dimensionless parameter $K = k V_0 / (g R)$ [5],

$$\begin{aligned} \varphi = & V_0 \left\{ \left(r + \frac{R^3}{2 r^2} \right) P_1(\cos \theta) + K \frac{R^4}{2 r^3} P_2(\cos \theta) - \right. \\ & \left. - K^2 \frac{9 R^3}{20 r^2} \left[P_1(\cos \theta) - \frac{3}{2} \left(\frac{R}{r} \right)^2 P_3(\cos \theta) \right] + \dots \right\}. \end{aligned} \quad (30)$$

If $c = 0$, then

$$k = k_* b^2, \quad p_2 = F/(b k_*^{1/2}),$$

and, from (29),

$$\varphi(r, \theta) = \varphi_0(r, \theta) + \frac{k_* b^2 V_0^2 R^3}{2gr^3} P_2(\cos \theta) + \dots, \quad (31)$$

therefore the terms in $K(= V_0 k_* b^2 (gR)^{-1})$ conicide, as it should to be expected. The same coincidence is found if we compare the term in K of φ_2 in [5] with the corresponding terms as it is pointed out by (28).

The expression of the velocity, up to terms in k_*^2 , can be computed from (14), (27) and (29). For the hydrodynamic action on the body we must use an appropriate expression for the porosity as a function of r and θ .

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EFFECT OF SUSPENDED PARTICLES ON THE EKMAN BOUNDARY LAYER

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Solutions are given for two problems on the motion of a dusty gas in the Ekman boundary layer over a rigid body, the bulk concentration of the suspended particles being small. In the first problem, the steady flow in the Ekman layer over a flat plate is considered when the plate moves with a uniform velocity parallel to itself. It is shown that the thickness of the layer decreases with increase in the mass concentration of the dust f . In the second problem the flow of a dusty gas in the semi-infinite region bounded by a plate is analyzed when the gas and the plate rotate like a solid body and additionally the plate executes nontorsional oscillations in its own plane. The flow exhibits a double boundary layer except in the case of resonance.

1. Introduction

Saffman [1] formulated the motion of a dusty gas in terms of a large number density $N(\vec{x}, t)$ or a cloud of undeformable spherical particles having single radius η , under the assumption that the bulk concentration of the particles is very small. However the density of the dust material is taken to be large compared with the gas density, so that the mass concentration of the dust f is an appreciable fraction of unity. Using this formulation, Michael and Miller [2] investigated the motion of a dusty gas occupying the semi-infinite space above a rigid plane when the plane moves parallel to itself (i) in simple harmonic motion, and (ii) impulsively started from rest with uniform velocity. Marble [3] studied the Blasius problem for a uniformly loaded, incompressible dusty medium moving parallel to a semi-infinite flat plate, in the case when the relative velocity between the fluid and the dust particles is very small. Zung [4] examined the flow due to the rotation of a circular disc in a quiescent medium with suspended particles and until now this remains the only solution for a dusty gas with a three-dimensional structure. A comprehensive review of the dynamics of dusty gases was recently given by Marble [3]. Little work seems to have been done on the flow of a dusty gas in a rotating medium although this problem has a bearing on the motion of aerosol suspension over the rotating earth. This provides the motivation for the present study. In this paper we have discussed the steady flow of a dusty gas in the Ekman layer over an infinite flat plate when the plate moves with a uniform velocity parallel to itself. We have also investigated the flow of a dusty gas in the semi-infinite region bounded by a plate when the fluid and the plate rotate like a solid body and additionally the plate performs non-torsional oscillations in its own plane.

2. Ekman layer on a moving plate

Consider the flow of a dusty gas over an infinite flat plate which moves parallel to itself relative to a frame of reference rotating with the gas with uniform angular velocity Ω . Since the dust particles are assumed very small, the Stokes law of resistance between the particles and the gas is appropriate, and the equations of motion in a rotating frame are

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{q} + \frac{kN}{\rho} (\vec{q}' - \vec{q}) - 2\vec{\Omega} \times \vec{q}, \quad (1)$$

$$Nm \left(\frac{\partial \vec{q}'}{\partial t} + (\vec{q} \cdot \nabla) \vec{q}' \right) = Nk(\vec{q} - \vec{q}') - 2mN\vec{\Omega} \times \vec{q}', \quad (2)$$

$$\nabla \cdot \vec{q} = 0, \quad (3)$$

$$\frac{\partial N}{\partial t} + \nabla \cdot (N\vec{q}') = 0 \quad (4)$$

Here $\vec{q}$, $\vec{q}'$ are the gas and dust velocities, the gas being assumed incompressible. Further m is the mass of a dust particle, K is the Stokes resistance coefficient $6\pi\mu\eta$ (μ and η being the viscosity of the clean gas and the radius of a particle respectively) and N is the number density of dust particles mentioned before. The other symbols have their usual meaning.

In the present problem we take z -axis along the axis of rotation so that the plate coincides with the plane $z = 0$. We further assume that the plate moves along x -axis with velocity U and the motion is steady. The horizontal homogeneity of the problem shows that the physical variables are functions of z only. If (u, v, w) and (u', v', w') are the components of $\vec{q}$ and $\vec{q}'$ respectively, equation (3) and the no-slip condition at the plate $z = 0$ imply that $w \equiv 0$ everywhere in the flow. Similarly equation (4) is satisfied by $w' \equiv 0$ and $N = N_0$, a constant throughout the motion. Equations (1) and (2) then give in the absence of any imposed pressure gradient

$$\nu \frac{d^2 u}{dz^2} + 2\Omega v + \frac{KN_0}{\rho} (u' - u) = 0, \quad (5)$$

$$\nu \frac{d^2 v}{dz^2} - 2\Omega u + \frac{KN_0}{\rho} (v' - v) = 0, \quad (6)$$

$$K(u - u') + 2 m\Omega v' = 0, \quad (7)$$

$$K(v - v') - 2 m\Omega u' = 0 \quad (8)$$

Equations (7) and (8) give

$$u' = \frac{K^2 u + 2m\Omega K v}{K^2 + 4m^2\Omega^2}, v' = \frac{K^2 v - 2m\Omega K u}{K^2 + 4m^2\Omega^2}. \quad (9)$$

Combining (5) and (6) and using (9), we get

$$\frac{d^2 F}{d\eta^2} - \left[i + \frac{mKN_0 (iK + 2m\Omega)}{\rho(K^2 + 4m^2\Omega^2)} \right] F = 0 \quad (10)$$

where

$$F(\eta) = \frac{u}{U} + i \frac{v}{U}, \quad \eta = z \left(\frac{2\Omega}{v} \right)^{\frac{1}{2}}. \quad (11)$$

Since $u = U$, $v = 0$ at $z = 0$ and $u \rightarrow 0$, $v \rightarrow 0$ as $z \rightarrow \infty$, the boundary conditions for (10) are

$$F(0) = 1, \quad F(\infty) = 0 \quad (12)$$

The solution of (10) satisfying (12) is

$$F(\eta) = \exp \left[- \left\{ \left(\frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha}{2} \right)^{\frac{1}{2}} + i \left(\frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha}{2} \right)^{\frac{1}{2}} \right\} \eta \right], \quad (13)$$

where

$$\alpha = \frac{2m^2\Omega K N_0}{\rho(K^2 + 4m^2\Omega^2)}, \quad \beta = 1 + \frac{mK^2 N_0}{\rho(K^2 + 4m^2\Omega^2)}. \quad (14)$$

Equations (11) and (13) give

$$\frac{u}{U} = \exp \left[- \left\{ \frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha}{2} \right\}^{\frac{1}{2}} \cdot \eta \right] \cdot \cos \left[\left\{ \frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha}{2} \right\}^{\frac{1}{2}} \cdot \eta \right], \quad (15)$$

$$\frac{v}{U} = -\exp\left[-\left\{\frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} + \alpha}{2}\right\}^{\frac{1}{2}} \cdot \eta\right] \cdot \sin\left[\left\{\frac{(\alpha^2 + \beta^2)^{\frac{1}{2}} - \alpha}{2}\right\}^{\frac{1}{2}} \cdot \eta\right]. \quad (16)$$

Equations (15) and (16) show that the velocity profile is in the form of an Ekman spiral (5), the dimensionless thickness of the Ekman layer being of the order of $[(\alpha^2 + \beta^2)^2 + \alpha]^{-\frac{1}{2}}$. Substituting from (14) we find that this thickness is of order.

$$\left[\left\{\frac{f(f+2)}{(1+G^2)} + 1\right\}^{\frac{1}{2}} + \frac{fG}{1+G^2}\right]^{-\frac{1}{2}}, \quad (17)$$

where

$$f = \frac{N_0 m}{\rho}, \quad G = \frac{2m\Omega}{K}. \quad (18)$$

It is clear from (17) that for fixed G , the thickness of the Ekman layer decreases with increase in the mass concentration parameter f of the dust particles. Skin friction coefficient c_f for the primary flow $u(z)$ is given from (15) as

$$c_f = \frac{-\mu\left(\frac{du}{dz}\right)_{z=0}}{\rho U^2} = \left[\left\{\frac{f(f+2)}{1+G^2} + 1\right\}^{\frac{1}{2}} + \frac{fG}{1+G^2}\right]^{\frac{1}{2}} \cdot E^{\frac{1}{2}}. \quad (19)$$

where E is the Ekman number $\nu\Omega/U^2$. Equation (19) shows that c_f increases with f and this is in keeping with the fact that increase in f causes thinning of the Ekman boundary layer. The skin-friction coefficient c'_f for the secondary flow is given by (16) as

$$c'_f = -\frac{\mu\left(\frac{dv}{dz}\right)_{z=0}}{\rho U^2} = \left[\left\{\frac{f(f+2)}{1+G^2} + 1\right\}^{\frac{1}{2}} - \frac{fG}{1+G^2}\right]^{\frac{1}{2}} \cdot E^{\frac{1}{2}}. \quad (20)$$

It can be easily shown that for fixed E and G , c'_f increases with f provided $G < 1$.

It may be seen from (19) that on the plate $z = 0$, u' and v' are given by $u' = K^2 U / (K^2 + 4m^2\Omega^2)$ and $v' = -2m\Omega KU / (K^2 + 4m^2\Omega^2)$. Thus it is interesting to note that the dust particles do not stick to the plate but move relative to it at an angle $\cot^{-1}(2m\Omega/K)$ with the negative y -axis. However such a velocity slip is compatible with the pre-

vious assumption that the bulk concentration of the dust particles is small since we can then imagine that the particles nearest to the plate will in general be several particle diameters away from it.

3. Ekman layer on an oscillating plate

Consider an infinite plate at $z = 0$ rotating in unison with a dusty gas in the region $z > 0$ with a uniform angular velocity Ω about z -axis. In addition to rotation, the plate executes non-torsional oscillations with velocity given by $U_1 (ae^{i\omega t} + be^{-i\omega t})$ in its own plane. Here a and b are complex and U_1 and ω are real. Horizontal homogeneity of the problem demands that the physical variables are functions of z and t only. Then as in the previous problem $w \equiv 0$, $w' \equiv 0$ everywhere in the flow and $N = N_0$, a constant throughout. Following the same notation as in Section 2, the equations of motion of the fluid are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + 2\Omega v + \frac{KN_0}{\rho} (u' - u), \quad (21)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\Omega u + \frac{KN_0}{\rho} (v' - v). \quad (22)$$

The equations of motion of the dust particles are

$$\frac{\partial u'}{\partial t} = \frac{K}{m} (u - u') + 2\Omega v', \quad (23)$$

$$\frac{\partial v'}{\partial t} = \frac{K}{m} (v - v') - 2\Omega u'. \quad (24)$$

The boundary conditions are

$$u + iv = U_1 (ae^{i\omega t} + be^{i\omega t}) \text{ at } z = 0, \quad (25)$$

$$u \rightarrow 0, \quad v \rightarrow 0 \quad \text{as } z \rightarrow \infty. \quad (26)$$

Let us introduce the dimensionless variables

$$q = \frac{u + iv}{U_1}, \quad \tau = \Omega t, \quad \eta = z \left(\frac{\Omega}{\nu} \right)^{\frac{1}{2}}. \quad (27)$$

Equations (21) and (22) can be combined as

$$\frac{\partial}{\partial t}(u + iv) = v \frac{\partial^2}{\partial z^2}(u + iv) - 2i\Omega(u + iv) + \frac{KN_0}{\rho} [(u' + iv') - (u + iv)], \quad (28)$$

while (23) and (24) give

$$\frac{\partial}{\partial t}(u' + iv') = \frac{K}{m} [(u + iv) - (u' + iv')] - 2i\Omega(u' + iv'). \quad (29)$$

Elimination of $u' + iv'$ from (28) and (29) gives after using (27)

$$\frac{\partial^2 q}{\partial \tau^2} - \frac{\partial^2 q}{\partial \tau \partial \eta^2} + (4i + F + G) \frac{\partial q}{\partial \tau} - (2i + G) \frac{\partial^2 q}{\partial \eta^2} + [2i(F + G) - 4]q = 0, \quad (30)$$

where

$$F = \frac{KN_0}{\rho\Omega}, \quad G = \frac{K}{m\Omega}. \quad (31)$$

The boundary conditions (25) and (26) become

$$q = ae^{i\sigma\tau} + be^{-i\sigma\tau} \text{ at } \eta = 0, \quad (32)$$

$$q \rightarrow 0 \text{ as } \eta \rightarrow \infty,$$

where $\sigma = \omega/\Omega$.
Assume

$$q = f_1(\eta)e^{i\sigma\tau} + f_2(\eta)e^{-i\sigma\tau} \quad (33)$$

Substituting (33) in (30) and equating coefficients of $e^{i\sigma\tau}$ and $e^{-i\sigma\tau}$, we have

$$f_1''(\eta) + \frac{(\sigma + 2)[(\sigma + 2) - i(F + G)]}{[G + (\sigma + 2)i]} f_1 = 0, \quad (34)$$

$$f_2''(\eta) + \frac{(2 - \sigma)[(2 - \sigma) - i(F + G)]}{[G + i(2 - \sigma)]} f_2 = 0 \quad (35)$$

The boundary conditions for $f_1(\eta)$ and $f_2(\eta)$ are obtained from (32) as

$$f_1(0) = a, \quad f_1(\infty) = 0, \quad f_2(0) = b, \quad f_2(\infty) = 0 \quad (36)$$

Thus the flow is characterized by the three dimensionless parameters F , G and σ .

The solutions of (34) and (35) satisfying (36) are

$$f_1(\eta) = ae^{-p_1[(\alpha_1 + \beta_1)^{\frac{1}{2}} + 1(\alpha_1 - \beta_1)^{\frac{1}{2}}]\eta}, \quad (37)$$

$$f_2(\eta) = be^{-p_2[(\alpha_2 + \beta_2)^{\frac{1}{2}} + 1(\alpha_2 - \beta_2)^{\frac{1}{2}}]\eta}, \quad (38)$$

where

$$p_1 = \left[\frac{2 + \sigma}{2\{G^2 + (2 + \sigma)^2\}} \right]^{\frac{1}{2}}, \quad p_2 = \left[\frac{(2 - \sigma)}{2\{G^2 + (2 - \sigma)^2\}} \right]^{\frac{1}{2}},$$

$$\alpha_1 = [F^2(2 + \sigma)^2 + \{(F + G)G + (2 + \sigma)^2\}^2]^{\frac{1}{2}}, \quad (39)$$

$$\alpha_2 = [F^2(2 - \sigma)^2 + \{(F + G)G + (2 - \sigma)^2\}^2]^{\frac{1}{2}},$$

$$\beta_1 = F(2 + \sigma), \quad \beta_2 = F(2 - \sigma).$$

It should be mentioned that the above solution is valid when $\sigma < 2$. It is interesting to note from (33), (37) and (38) that the non-torsional oscillations of the plate give rise to a double boundary layer structure near the plate, the dimensionless thicknesses of these two layers being of order $p_1^{-1}(\alpha_1 + \beta_1)^{-\frac{1}{2}}$ and $p_2^{-1}(\alpha_2 + \beta_2)^{-\frac{1}{2}}$. Denoting these by δ_1 and δ_2 respectively, we have from (39),

$$\delta_1 = \left[\frac{2\{G^2 + (2 + \sigma)^2\}}{2 + \sigma} \right]^{\frac{1}{2}} \cdot [(F^2(2 + \sigma)^2 + \{(F + G)G + (2 + \sigma)^2\}^2)^{\frac{1}{2}} + F(2 + \sigma)]^{-\frac{1}{2}},$$

... (40)

$$\delta_2 = \left[\frac{2\{G^2 + (2 - \sigma)^2\}}{2 - \sigma} \right]^{\frac{1}{2}} \cdot [(F^2(2 - \sigma)^2 + \{(F + G)G + (2 - \sigma)^2\}^2)^{\frac{1}{2}} + F(2 - \sigma)]^{-\frac{1}{2}}$$

(41)

For fixed G and σ , both δ_1 and δ_2 clearly decrease with increase in F/G . But from (31), F/G can be written as $N_0 m/\rho$ which is the mass concentration parameter f introduced in Section 2. Thus both mass the boundary layer thicknesses decrease with increase in f , so that the shear stresses at the wall increase with increase in f . The solution for $\sigma > 2$ can be found in a similar manner.

When $\sigma = 2$, equation (35) gives $f_2''(\eta) = 0$ and there is no solution of this equation satisfying $f_2(0) = b$, $f_2(\infty) = 0$. In this case one of the boundary layers blows off. Thus when $\omega = 2\Omega$, resonance occurs and the shear oscillations no longer remain confined to the plate.

4. Discussion

It should be mentioned that in our analysis the dust particles are assumed to be non-interacting so that all particles in a local volume have the same velocity vector and temperature. This is due to the fact that in addition to collisions, velocity and temperature could be random only because of Brownian motion in the gas, for which the dust particles are too large, or due to random initial conditions, which would be damped as soon as the motion starts.

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REMARQUE CONCERNANT L'APPROXIMATION DES SOLUTIONS DES ÉQUATIONS FONCTIONNELLES LINÉAIRES

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Dans ce travail on obtient un résultat d'existence et d'approximation des solutions d'une équation fonctionnelle linéaire à l'aide d'une suite de solutions de certaines équations autoadjointes dans certains espaces de Hilbert.

Soit E un espace de Hilbert, F un espace localement convexe, E' , F' leurs duaux, $A \in \mathcal{L}(E, F)$, A' le transposé de A , $\langle, \rangle$ la parenthèse de dualité entre un espace et son dual, $(,)$ le produit scalaire dans un espace de Hilbert.

Comme on le sait, l'équation

$$Au = f, \quad (1)$$

ou $f \in F$, admet des solutions $u \in E$ si et seulement s'il existe une constante $c(f)$ pour laquelle est vérifiée l'inégalité

$$|\langle f, v \rangle| \leq c(f) \|A'v\|_{E'}, \quad (\forall) v \in F'. \quad (2)$$

Dans l'hypothèse (2), l'équation (1) admet une seule solution $u \in E$ pour laquelle

$$\|u\|_E = \max_{v \in F'} \frac{|\langle f, v \rangle|}{\|A'v\|_{E'}} = M(f). \quad (3)$$

Cette solution (appelée solution extrémale) a la norme minimale et est orthogonale sur $\text{Ker } A$.

On supposera qu'il existe une suite G_n , $n = 1, 2, \dots$, d'espaces de Hilbert et les applications $\sigma_n \in \mathcal{L}(F, G_n)$, telles que si G'_n est le dual de G_n et σ'_n la transposée de σ_n , les espaces $\sigma'_n(G'_n)$ forment une suite croissante et $\bigcup_{n=1}^{\infty} \sigma'_n(G'_n)$ soit dense dans F' pour la topologie faible.

Soient $A_n = \sigma_n \circ A \in \mathcal{L}(E, G_n)$, $f_n = \sigma_n(f)$ et on considère les équations

$$A_n u_n = f_n, \quad n = 1, 2, \dots \quad (1')$$

Proposition 1. Si les nombres

$$M_n(f) = \max_{v' \in G'_n} \frac{|\langle f_n, v' \rangle|}{\|A'_n v'\|_{E'}}, \quad n = 1, 2, \dots \quad (3')$$

forment une suite bornée, les équations (1) et (1') admettent des solutions et les solutions extrémales des équations (1') convergent dans E vers la solution extrémale de l'équation (1).

Démonstration. Conformément aux affirmations antérieures, les équations (1') admettent les solutions extrémales u_n , pour lesquelles $\|u_n\|_E = M_n(f)$. Si $v' \in G'_n$ et $v = \sigma'_n(v')$, alors $\langle f, v \rangle = \langle \sigma_n(f), v' \rangle = \langle f_n, v' \rangle$, et $A'v = A'_n v'$; en tenant compte de l'hypothèse et du fait que $\bigcup_{n=1}^{\infty} \sigma'_n(G'_n)$ est dense dans F' , il résulte que l'expression (3) est finie, d'où il résulte l'existence d'une solution extrémale pour l'équation (1), avec $\|u\|_E = M(f)$. La suite $\{M_n(f)\}$ est croissante et $\lim_{n \rightarrow \infty} M_n(f) = M(f)$.

La suite $\{u_n\}_{n=1}^{\infty}$ étant bornée dans E , il existe $u_* \in E$ et une sous-suite $\{u_{n_k}\}$ faiblement convergente vers u_* ; évidemment, $\|u_*\|_E \leq \|u\|_E$. Montrons que u_* vérifie (1).

Puisque $\sigma_{n_k}(Au_{n_k} - f) = 0$, si $v' \in G'_{n_k}$ et $v = \sigma'_{n_k}(v')$, on aura $\langle Au_{n_k} - f, v \rangle = 0$. Soient $\varepsilon > 0$ et $w \in \sigma'_p(G'_p)$. De l'égalité $\langle A(u_{n_k} - u_*), w \rangle = \langle u_{n_k} - u_*, A'w \rangle$ et du fait que $u_{n_k} - u_*$ tend faiblement vers zéro, il résulte qu'on peut prendre n_k suffisamment grand pour que $|\langle A(u_{n_k} - u_*), w \rangle| < \varepsilon$. Alors,

$$|\langle Au_* - f, w \rangle| \leq |\langle Au_{n_k} - f, w \rangle| + \varepsilon = \varepsilon,$$

car pour $n_k > p$, $w \in \sigma'_{n_k}(G'_{n_k})$.

Il résulte $\langle Au_* - f, w \rangle = 0$ ($\forall$) $w \in \bigcup_{p=1}^{\infty} \sigma'_p(G'_p)$, donc $Au_* = f$. u_* étant solution extrémale, $u_* = u$ et la suite entière $\{u_n\}$ tend faiblement vers u . Puisque $\lim_{n \rightarrow \infty} \|u_n\|_E = \|u\|_E$, il résulte que $\lim_{n \rightarrow \infty} u_n = u$ dans E .

q.e.d.

Evidemment, la condition sur la suite $\{M_n(f)\}$ est aussi nécessaire pour l'existence des solutions de l'équation (1).

Dans certains cas, l'équation (1') peut être réduite à l'étude d'un opérateur autoadjoint dans l'espace de Hilbert G_n . En effet, soit A_n^* l'adjoint de A , c'est-à-dire $A_n^* \in \mathcal{L}(G_n, E)$ et

$$(A_n u, v) = (u, A_n^* v), (\forall) u \in E, v \in G_n.$$

Alors $B_n = A_n A_n^* \in \mathcal{L}(G_n, G_n)$ et il est autoadjoint dans G_n . En cherchant des solutions de l'équation (1') de la forme

$$u_n = A_n^* v_n, \quad (4)$$

on arrive à l'équation

$$B_n v_n = f_n. \quad (5)$$

Si v_n est une solution quelconque de l'équation (5) et $w_n \in \text{Ker } A_n$, alors $(u_n, w_n) = (v_n, A_n w_n) = 0$, c'est-à-dire u_n donné par (4) est une solution extrémale de l'équation (1'). On remarque que $f_n \perp \text{Ker } B_n$, car pour $w \in \text{Ker } B_n$, $\|A_n^* w\|^2 = (B_n w, w) = 0$, donc $w \in \text{Ker } A_n^*$ et $f_n \perp \text{Ker } A_n^*$ dans l'hypothèse que l'équation (1') a des solutions. Alors, si l'image $R(B_n)$ de l'opérateur B_n est fermée, l'équation (5) a des solutions (ce fait a lieu, par exemple, si G_n est de dimension finie).

Dans ce cas B_n est un isomorphisme de l'espace $(\text{Ker } B_n)^\perp$ sur $R(B_n)$ et on a démontré que pour $f \perp \text{Ker } A'$ a lieu la

Proposition 2. Si $R(B_n)$ sont des ensembles fermés dans G_n , $n = 1, 2, \dots$, la condition nécessaire et suffisante d'existence des solutions de l'équation (1) est l'existence d'une constante $c(f)$ telle que

$$(B_n^{-1} f_n, f_n) \leq c(f), \quad n = 1, 2, \dots \quad (6)$$

Dans ce cas les fonctions (4) convergent dans E vers la solution extrémale de l'équation (1).

En effet, il suffit de remarquer que $\|u_n\|_E^2 = (B_n^{-1} f_n, f_n)$.

Dans les hypothèses précédentes, $R(B_n) = (\text{Ker } B_n)^\perp$, donc B_n est un opérateur autoadjoint dans l'espace de Hilbert $(\text{Ker } B_n)^\perp$ et isomorphisme du même espace sur lui-même; soit $\sigma(B_n)$ le spectre de B_n , qui ne contiendra pas le point $\lambda = 0$. La norme de B_n^{-1} dans l'espace $(\text{Ker } B_n)^\perp$ sera donnée par l'expression

$$\|B_n^{-1}\| = \max_{\lambda \in \sigma(B_n)} \frac{1}{\lambda}. \quad (7)$$

Il résulte alors pour $f \perp \text{Ker } A'$, la

Proposition 3. Si $R(B_n)$ sont des ensembles fermés dans G_n et la suite $\{\|\sigma_n(f)\|\}$ est bornée (en particulier si les applications σ_n sont uniformément bornées), une condition suffisante d'existence des solutions de l'équation (1) est l'inégalité

$$\max_{\lambda \in \sigma(B_n)} \frac{1}{\lambda} \leq C, \quad n = 1, 2, \dots, \quad (7')$$

avec C constante convenable.

Un exemple très simple et bien connu où les constructions précédentes s'appliquent est celui des systèmes algébriques linéaires de dimension infinie

$$\sum_{j=1}^{\infty} a_{ij} x_j = f_i, \quad i = 1, 2, \dots \quad (8)$$

ou a_{ij} , x_j , $f_i \in C$. On désigne par $f = \{f_i\}_{i=1}^\infty$ un élément quelconque de l'espace F de toutes les suites, muni de la topologie de convergence de chaque terme (il s'agit donc d'un espace de Montel); E sera l'espace l_2 de toutes les suites $u = \{x_j\}_{j=1}^\infty$ telles que $\|u\|^2 = \sum |x_j|^2 < +\infty$. En supposant que pour tout, i , $\{a_{ij}\}_{j=1}^\infty \in l_2$, l'opérateur A de matrice (a_{ij}) sera dans $\mathcal{L}(E, F)$. On peut prendre $G_n = C^n$ et $\sigma_n(f) = \{f_1, \dots, f_n\}$, donc $\sigma_n \in \mathcal{L}(F, G_n)$, étant uniformément bornées par rapport à n sur $l_2 \hookrightarrow F$. Les opérateurs A_n seront définis par les matrices $(a_{ij})_{1 \leq i \leq n, 1 \leq j < \infty}$ et B_n par $(b_{ij})_{1 \leq i, j \leq n}$,

ou $b_{ij} = \sum_{k=1}^\infty a_{ik} \bar{a}_{kj}$. B_n , agissant dans un espace de dimension finie, aura l'image fermée et les spectre constitué par un nombre fini de valeurs propres réelles ≥ 0 .

S'il existe une constante strictement positive (indépendante de n) majorée par toutes ces valeurs propres (sauf zéro), le système (8) possède au moins une solution dans l_2 (qui puisse être approchée à l'aide des solutions de systèmes finis) pour tout $f \in l_2$, $f \perp \text{Ker } A'$.

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A REPRESENTATION THEOREM FOR THE SPACE OF HERMITIAN ELEMENTS IN A BANACH ALGEBRA

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Let A be a complex Banach algebra with identity. Let A' denote the dual space of A , i.e. the Banach space of all continuous linear functionals on A . Let $S = S(A)$ be the set of all states on A , i.e.

$$S = \{s \in A' \mid s(1) = 1 = \|s\|\}$$

and let $P = P(A)$ be the set of all pure states on A , i.e. the set of all extreme states.

Recall that an element $x \in A$ is called Hermitian if

$$s \in S \Rightarrow s(x) \in \mathbb{R}.$$

We denote the set of all Hermitian elements of A by $H(A)$. It is clear from the definition that $H(A)$ is a closed linear subspace of A .

We recall also that

$$x \in A \Rightarrow \rho(x) \leq v(x) \leq \|x\|$$

$$x \in H(A) \Rightarrow \|x\| = \rho(x) \quad (\text{Sinclair's theorem [8]})$$

where $\rho(x)$ is the spectral radius of x and $v(x)$ is the numerical radius of x , i.e.

$$v(x) = \sup_{s \in S} |s(x)|.$$

The linear space of all real affine and continuous functions on S will be denoted by $\mathcal{A}(S)$.

To prove the following theorem we shall need the following usually fact of convexity theory:

If S is a compact convex subset of a locally convex Hausdorff space F and μ a probability measure on S , i.e. a measure with the property $\mu(I) = 1 = \|\mu\|$, then there is a unique $s \in S$ such that

$$u \in \mathcal{A}(S) \Rightarrow \mu(u) = u(s),$$

called the barycenter of μ .

Theorem 1. If A is a complex Banach algebra with identity and $H(A)$ separates points of S , i.e. $s_1(h) = s_2(h)$ for all $h \in H(A)$ implies $s_1 = s_2$ ($s_1, s_2 \in S$), then $H(A)$ is isometrically isomorphic with $\mathcal{A}(S)$.

For each $x \in H(A)$ we define the following function :

$$\hat{x} : S \rightarrow R, \hat{x}(s) = s(x).$$

Clearly

$$x \rightarrow \hat{x}$$

is a homomorphism of $H(A)$ into $\mathcal{A}(S)$. Moreover, from the Sinclair's theorem, follows that $\|x\| = \|\hat{x}\|$, hence that the range of $H(A)$ by $x \rightarrow \hat{x}$ which will be denoted by $\widehat{H(A)}$, is a closed linear subspace of $\mathcal{A}(S)$ which separates points of S and contains the constant functions. By a standard argument we deduce that $\widehat{H(A)} = \mathcal{A}(S)$. For this, let f be a continuous linear functional on $\mathcal{A}(S)$ such that $f|_{\widehat{H(A)}} = 0$. By the Hahn-Banach theorem, there exists a real measure on S such that $\mu|_{\widehat{H(A)}} = 0$. Let μ_1 and μ_2 two positives measures on S such that $\mu = \mu_1 - \mu_2$. Since $\mu(1) = 0$, we can assume that $\mu_1(1) = \mu_2(1) = 1$. Let now s_1 the barycenter of measure μ_1 and s_2 the barycenter of measure μ_2 . Thus, for each $u \in \widehat{H(A)}$, we have

$$0 = \mu(u) = \mu_1(u) - \mu_2(u) = u(s_1) - u(s_2)$$

and so, from the fact that $\widehat{H(A)}$ separates points of S , follows that $s_1 = s_2$. Therefore, for each $u \in \mathcal{A}(S)$,

$$f(u) = \mu(u) = \mu_1(u) - \mu_2(u) = u(s_1) - u(s_2) = 0,$$

and from this follows that $\widehat{H(A)} = \mathcal{A}(S)$.

Corollary 2. If A is a complex Banach algebra with identity such that $A = H(A) + iH(A)$, then $H(A)$ is isometrically isomorphic with $\mathcal{A}(S)$.

Corollary 3. If A is a C^* -algebra with identity, then $H(A)$ is isometrically isomorphic with $\mathcal{A}(S)$.

Follows from the fact that

$$x \in H(A) \Leftrightarrow x^* = x \text{ (see [4]).}$$

Corollary 4. If A is a C^* -algebra with identity and $\mathcal{A}_c(S)$ the space of all complex affine and continuous functions on S , then the mapping $x \rightarrow \hat{x}$ is a bicontinuous isomorphism of A onto $\mathcal{A}_c(S)$ with the property

$$h = h^* \Rightarrow \hat{h} = \overline{\hat{h}} \text{ and } \|h\| = \|\hat{h}\|.$$

Since $A = H(A) + iH(A)$, $\mathcal{A}_C(S) = \mathcal{A}(S) + i\mathcal{A}(S)$ by the preceding corollary follows that the mapping $x \rightarrow x$ is onto. From the fact that

$$x \in A \Rightarrow \frac{\|x\|}{2} \leq v(x) \leq \|x\|$$

follows that the mapping $x \rightarrow \hat{x}$ is a bicontinuous isomorphism.

Corollary 5. If A is a commutative C^* -algebra with identity and $\mathcal{A}_C(S)$ the space of all complex affine and continuous functions on S then A is isometrically isomorphic with $\mathcal{A}_C(S)$.

If A is a commutative C^* -algebra with identity then it is clear that

$$x \in A \Rightarrow \|x\| = \rho(x).$$

Since

$$x \in A \Rightarrow \rho(x) \leq v(x) \leq \|x\|$$

it follows that

$$x \in A \Rightarrow \|x\| = v(x)$$

and so

$$x \in A \Rightarrow \|x\| = \|\hat{x}\|.$$

Corollary 6. (Gelfand-Naimark) Let A be a commutative C^* -algebra with identity and $C(M)$ the set of all complex continuous functions on the set M of all multiplicative linear functionals on A . Then the mapping $x \rightarrow \hat{x}|_M$ is an isometric star isomorphism of A onto $C(M)$.

From the fact that $\text{co } P = S$ and the preceding corollary follows that

$$x \in A \Rightarrow \|\hat{x}|_P\| = \|x\| = \|\hat{x}\|.$$

On the other hand, since A is a commutative C^* -algebra with identity, we have $P = M$ (see [7]) and so

$$x \in A \Rightarrow \|\hat{x}|_M\| = \|\hat{x}\| = \|x\|.$$

Hence, by the Stone-Weierstrass theorem follows that the mapping $x \rightarrow \hat{x}|_M$ is a bijection.

Let now H a Hilbert space and $B(H)$ the algebra of all bounded linear operators on H .

For any $h \in H$ with $\|h\| = 1$, the mapping

$$p_h : B(H) \rightarrow C$$

defined by

$$p_h(x) = \langle xh, h \rangle$$

is a state on $B(H)$. Such a state is called a *vector state*. We denote by P_H the set of all vector states and by $P = P(B(H))$. It is known that $P_H \subset P$ (see [4]). Also, can be extracted from the proof of a theorem of Lumer ([5], theorem 11), a proof for the inclusion $P \subset \bar{P}_H$, where by $\bar{P}_H$ denoted the closure of P in A' (see also [4], 11.6.1.). We shall give here a proof for the relation $P \subset \bar{P}_H$, using the corollary 3. For completeness we give also, a proof for the relation $P_H \subset P$.

For the proof we need some facts. We recall it. Let X be a compact Hausdorff space and $\mathcal{L}$ a real linear space of continuous functions on X which separates points of X and contains the constant functions. A closed subset F of X is called a *boundary of X with respects to $\mathcal{L}$* if

$$u \in \mathcal{L}, u|_F \geq 0 \Rightarrow u \geq 0 \text{ on } X.$$

A probability measure on X is called a *representing measure* for a point $x \in X$ with respect to $\mathcal{L}$ if

$$u \in \mathcal{L} \Rightarrow u(x) = \mu(u).$$

The set of all representing measures for x will be denoted by M_x . The unit mass ϵ_x at x always belongs to M_x . The set ∂X of all points $x \in X$ for which $M_x = \{\epsilon_x\}$ is called the *Choquet boundary* of X with respect to $\mathcal{L}$. It is known that $\bar{\partial X}$ is exactly the intersection of all closed boundaries of X with respect to $\mathcal{L}$ ([1]).

Recall also that a positive operator on H is a bounded linear operator on H with the property $\langle xh, h \rangle \geq 0$ for all $h \in H$. An element $x \in B(H)$ is positive if and only if $s(x) \geq 0$ for all states s on $B(H)$.

Theorem 7. For any Hilbert space H we have

$$P_H \subset P \subset \bar{P}_H.$$

Let $p_h \in P_H$. Suppose that $p_h = \frac{s_1 + s_2}{2}$ with $s_1, s_2 \in S$. Then

$$2p_h - s_1 = s_2 \Rightarrow s_1 \leq 2p_h.$$

We can easily see that $H = \{xh | x \in B(H)\}$.

Let $F: H \times H \rightarrow C$ defined by $F(\xi, \eta) = s_1(y^*x)$, where $y, x \in B(H)$ and $xh = \xi, yh = \eta$.

The definition of F does not depend on the special choice of x and y . Indeed, let $\xi = xh$ and $\xi = x'h$. Then $(x - x')h = 0$ and hence

$$p_h[(x - x')^*(x - x')] = \|(x - x')h\|^2 = 0$$

and from the fact that $s_1 \leq 2p_h$ follows that

$$s_1[(x - x')^*(x - x')] = 0.$$

Now, using the Schwartz inequality

$$|s(zt)|^2 \leq s(zz^*)s(t^*t)$$

which is verified for any $s \in S$ and any z, t in a C^* -algebra, we deduce that

$$s_1[y^*(x - x')] = 0, \quad s_1(y^*x) = s_1(y^*x').$$

By the same reasoning we prove that F does not depend on the choice of y .

The mapping F is a bounded bilinear form on H , since

$$\begin{aligned} |F(\xi, \eta)| &= |s_1(y^*x)| \leq s_1(x^*x)^{1/2} s_1(y^*y)^{1/2} \leq \\ &\leq \sqrt{2p_h}(x^*x)^{1/2} \sqrt{2p_h}(y^*y)^{1/2} = 2\|xh\| \|yh\| = 2\|\xi\| \|\eta\|. \end{aligned}$$

Hence there exists $a \in B(H)$ such that

$$F(\xi, \eta) = \langle a\xi, \eta \rangle$$

From the fact that $\langle a\xi, \xi \rangle = F(\xi, \xi) = s_1(x^*x) \geq 0$, follows that a is a positive operator.

We shall show that $ax = xa$ for any $x \in B(H)$ and so that $a = \lambda I$ with $\lambda \geq 0$.

We have

$$\langle ax\xi, \eta \rangle = \langle axx'h, y'h \rangle = F(xx'h, y'h) = s_1(y'^*xx'),$$

$$\langle ax\xi, \eta \rangle = \langle a\xi, x^*\eta \rangle = \langle ax'h, x^*y'h \rangle =$$

$$= F(x'h, x^*y'h) = s_1[(x^*y')^*x'] = s_1(y'^*xx')$$

and so

$$\langle ax\xi, \eta \rangle = \langle xa\xi, \eta \rangle$$

for any $\xi, \eta \in H$, which prove that $ax = xa$.

Therefore

$$s_1(x) = s(1^*x) = F(xh, h) = \langle \lambda xh, h \rangle = \lambda p_h(x)$$

i.e. $s_1 = \lambda p_h$. From this fact follows that $\lambda = 1$ and hence that $s_1 = p_h = s_2$, i.e. $p_h \in P$, since $s_1(1) = 1$ and $p(1) = 1$.

We shall prove, now, that $P \subset \bar{P}_H$. From the fact that $P_H \subset S$ follows that $\bar{P}_H \subset \mathcal{A}(S)$. Let $u \in \mathcal{A}(S)$, $u \geq 0$ on $\bar{P}_H$. From the corollary 3 follows that there exists $x \in B(H)$ such that $u = \hat{x}$. Since u is positive

on P_H then x is a positive operator on H and so $s(x) \geq 0$ for any $s \in S$. This means that u is positive on S and hence that P is a closed boundary of S with respect to $\mathcal{A}(S)$. Therefore $\partial S \subset \bar{P}_H$.

On the other hand, from the Bauer's characterisation for extreme points of a compact convex subset of a locally convex Hausdorff space, follows that $\partial S = P$ and so the relation $P \subset \bar{P}_H$ is proved.

Remark. It is not true in general that $P_H = P$.

Let l^2 be the space of square-summable sequences and

$$U : l^2 \rightarrow l^2$$

defined by

$$U(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$$

It is easy to see that $U = 1$. From the Schwartz inequality follows that

$$h \in l^2, \|h\| = 1 \Rightarrow |\langle Uh, h \rangle| \leq 1.$$

If $|\langle Uh, h \rangle| = 1$ then $Uh = \lambda h$ and this contradicts the definition of U .

From the preceding considerations follows that

$$W(U) \subset \{z \mid |z| < 1\},$$

where $W(U)$ is the numerical range of U , i.e. the set

$$\{|\langle Uh, h \rangle| \mid h \in l^2, \|h\| = 1\}.$$

On the other hand, it is well known that $\pi(U) = \{z \in C \mid |z| = 1\}$ and so there exists a pure state p such that $p(U) = 1$ (see [6], theorem 4). Since $1 \notin W(U)$, follows that $p \notin P_H$.

Corollary 8. For any Hilbert space H we have

$$\overline{\text{co}} P_H = S,$$

where $\overline{\text{co}} P_H$ denotes the closed convex hull of the set P_H .

Follows from the preceding theorem and the Krein-Milman theorem.

Corollary 9. Let $(x_1, \dots, x_n)$ be a finite system of elements of $B(H)$.

Then

$$\overline{\text{co}} W(x_1, \dots, x_n) = V(x_1, \dots, x_n)$$

where

$$W(x_1, \dots, x_n) = : \{(\langle x_1 h, h \rangle, \dots, \langle x_n h, h \rangle) \mid h \in H, \|h\| = 1\}$$

$$V(x_1, \dots, x_n) = : \{(s(x_1), \dots, s(x_n)) \mid s \in S\}.$$

Corollary 10. If $(x, \dots, x_n)$ is a *dfinite* system of normal elements of $B(H)$ then

$$\overline{W(x_1, \dots, x_n)} = V(x_1, \dots, x_n)$$

It is a consequence of the preceding corollary and a theorem of Dash ([3]), which states that the joint numerical range of a finite system of normal operators on a Hilbert space, i.e. the set $W(x_1, \dots, x_n)$, is a convex set.

Corollary 11. (Lumer [5], Berberian and Orland [2], Stampfli and Williams [8]) For any $x \in B(H)$ we have

$$\overline{W(x)} = V(x).$$

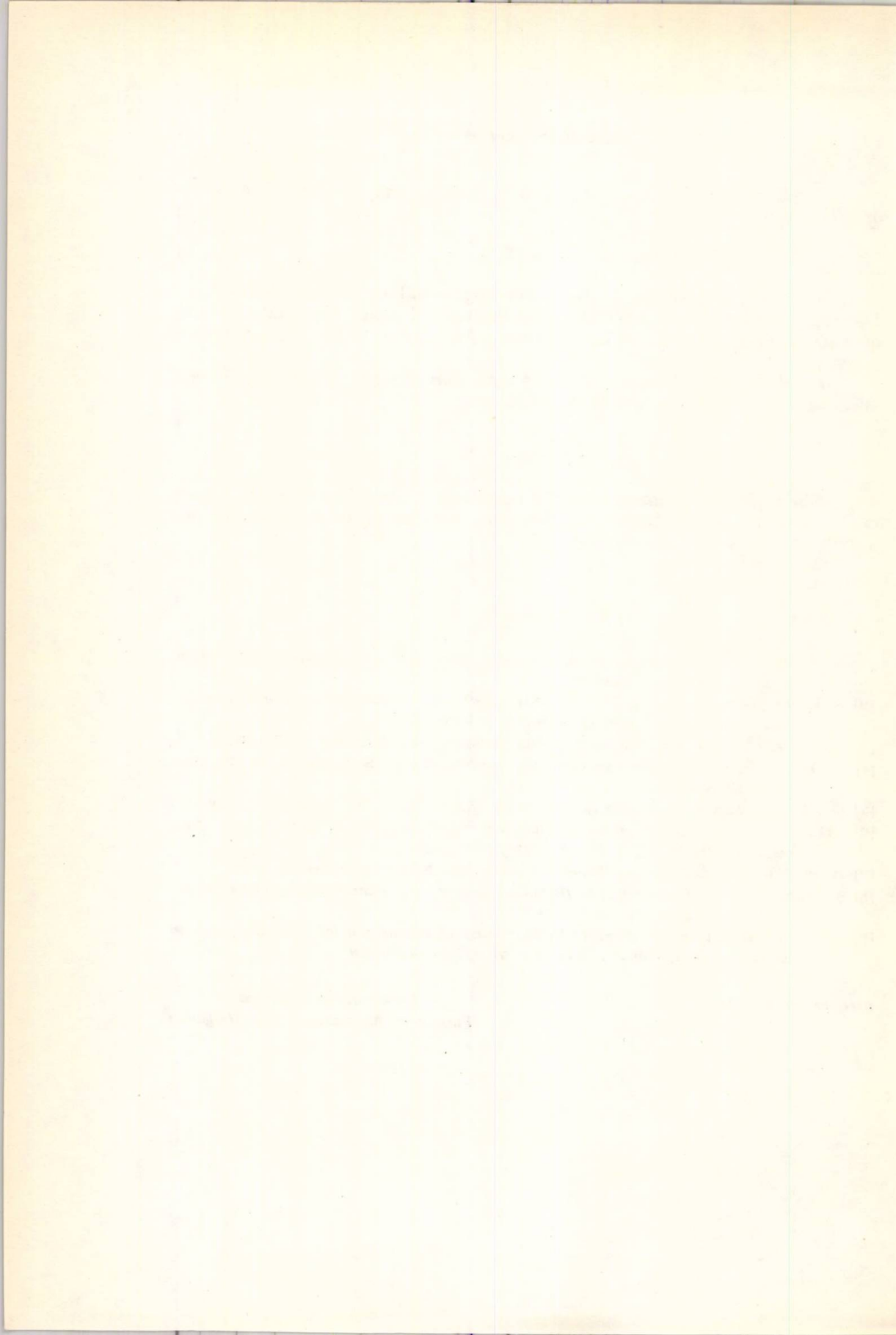
The assertion follows from the corollary 9 using the Toeplitz-Hausdorff theorem which states that the numerical range of an operator on a Hilbert space is a convex set.

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SUR QUELQUES RÉSULTATS NUMÉRIQUES POUR LES TRAJECTOIRES DE LA MASSE NULLE DANS LE SYSTÈME TERRE-LUNE

MARIA ELENA PITICU

1. Introduction

Nous considérons le cas plan du problème restreint. Soient $P_1(-m, 0)$, $P_2(M, 0)$ les positions des masses finies, M et m , dans le système d'axes entraînés. Puisque P_1P_2 est l'unité de longueur et nous prenons en plus $T = 2\pi$ (T représente la période du mouvement circulaire), il en résulte $k = 1$ (k représente la constante de l'attraction universelle), de sorte que les équations du mouvement de la masse nulle sont :

$$\begin{cases} \ddot{x} - 2\dot{y} = V_x \\ \ddot{y} + 2\dot{x} = V_y \end{cases} \quad (1)$$

où l'on a

$$\begin{cases} V = \frac{M}{R} + \frac{m}{r} + \frac{x^2 + y^2}{2} \\ R^2 = (x + m)^2 + y^2 \\ r^2 = (x - M)^2 + y^2 \end{cases} \quad (2)$$

Le système différentiel (1) admet l'intégrale de Jacobi

$$\dot{x}^2 + \dot{y}^2 = 2(V + h) \quad (3)$$

2. Le calcul des trajectoires

Nous avons considéré le cas du système Terre-Lune, dans lequel le rapport des masses finies est $M/m = 81,45$ et nous avons intégré numériquement le système différentiel (1) à l'aide de la méthode Runge-Kutta.

Le domaine du mouvement de la masse nulle nous l'avons précisé par la valeur de la constante h de l'intégrale première (3). Le pas a été choisi de sorte que pour chaque position calculée, la constante h de l'intégrale (3) doit rester invariable jusqu'à la cinquième décimale inclusivement. Les calculs ont été effectués à l'aide du calculateur IBM-360 de l'Université de Bucarest.

Les trajectoires ont été calculées pour les intervalles de temps τ de 32–80 jours, les positions et les vitesses de la masse nulle étant données avec 7 décimales exactes.

Nous avons considéré deux domaines du mouvement de la masse nulle correspondant aux valeurs $h_1 = -1,59406$ et $h_2 = -1,50606$ et nous avons calculé six trajectoires qui au moment initial t_0 sont orthogonales à la droite Terre-Lune, respectivement aux points :

$$x_0^1 = 0,11717; \quad x_0^2 = 0,24217; \quad x_0^3 = 0,36717; \quad x_0^4 = 0,49217$$

$$x_0^5 = 0,61717; \quad x_0^6 = 0,74217$$

conformément aux figures : 1, 2, 3, ..., 12.

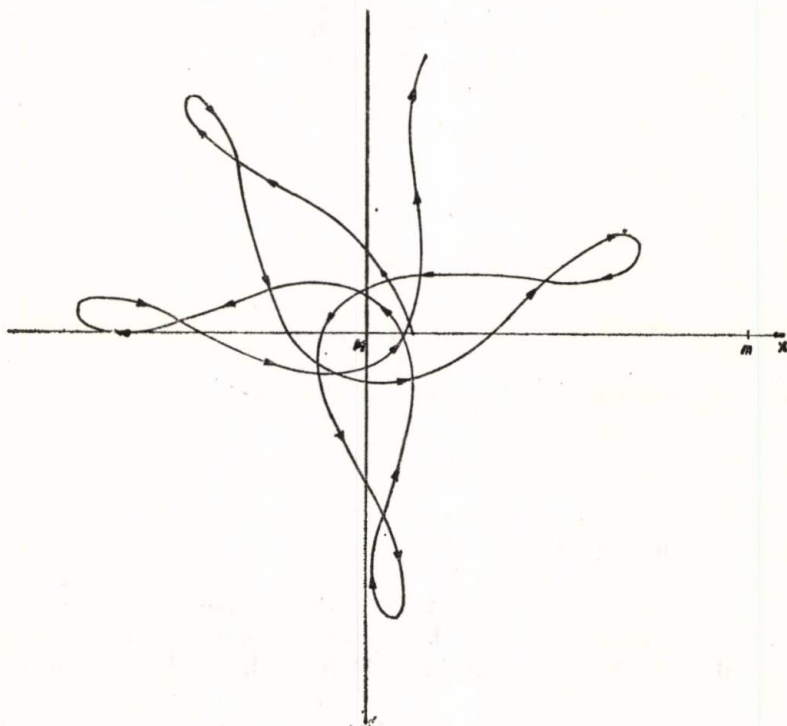


Fig. 1 $x_0 = 0,11717$ $h = -1,59406$ $\tau = 32$ jours

Les points de départ de toutes ces trajectoires sont situés à une distance du centre de la Lune plus grande que 66.000 km. En général, la masse nulle décrit des trajectoires qui contournent la Terre.

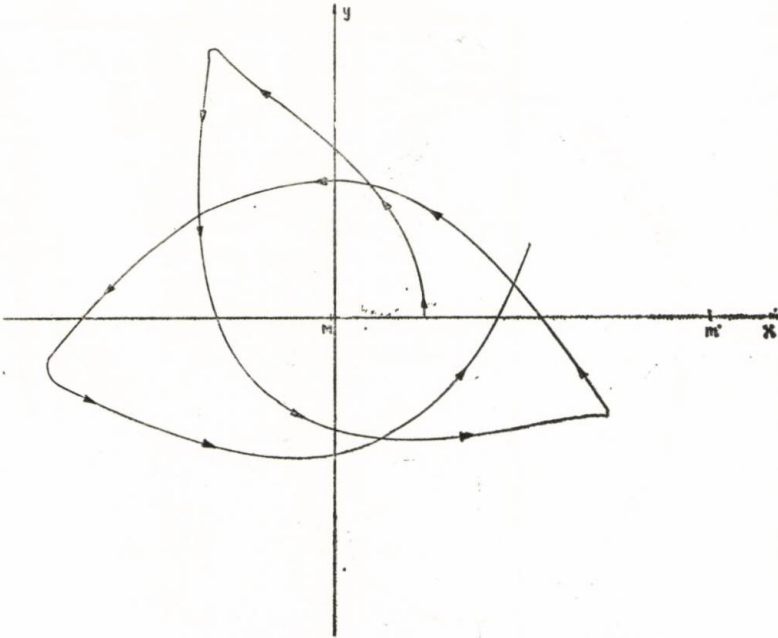


Fig. 2 $x_0 = 0,24217$ $h = -1,59406$ $\tau = 32$ jours

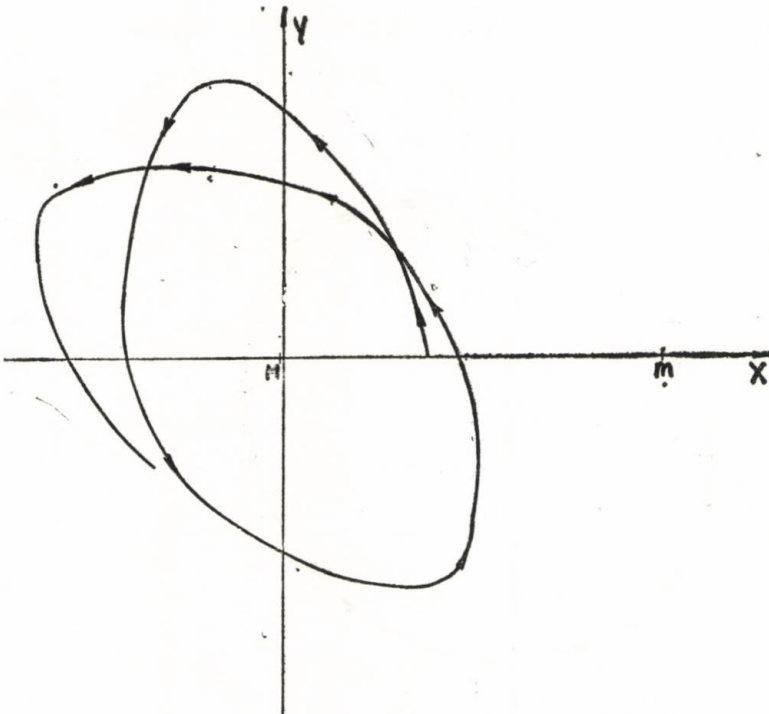


Fig. 3 $x_0 = 0,36717$ $h = -1,59406$ $\tau = 32$ jours

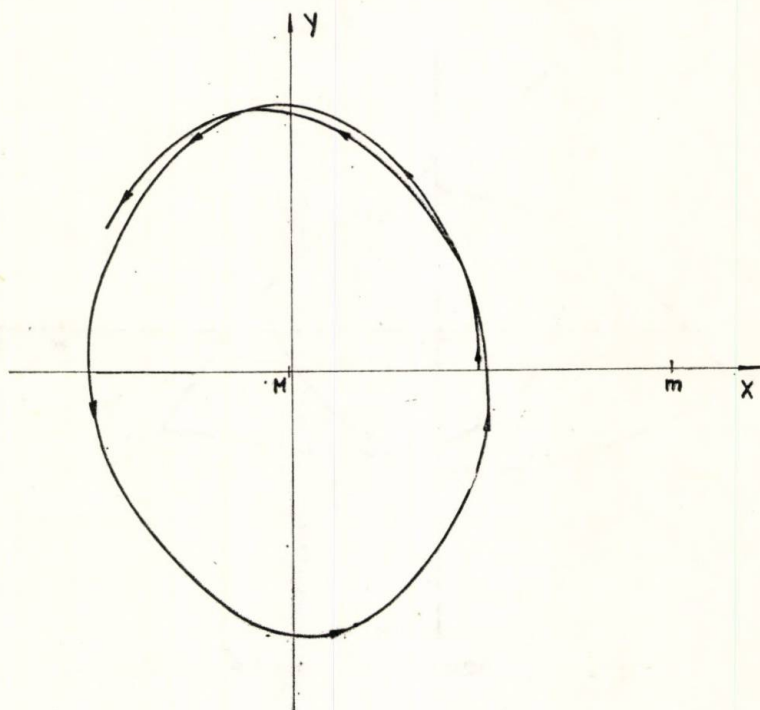


Fig. 4 $x_0 = 0,49217$ $h = -1,59406$ $\tau = 32$ jours

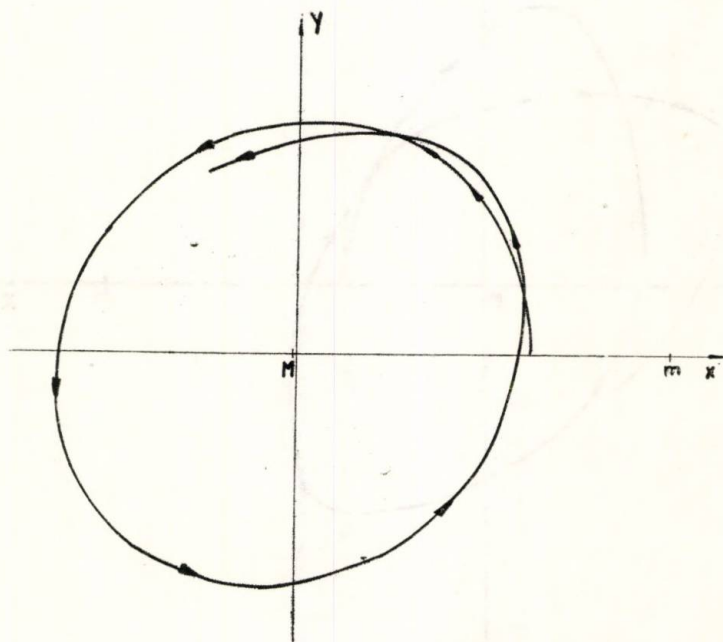


Fig. 5 $x_0 = 0,61717$ $h = -1,59406$ $\tau = 32$ jours

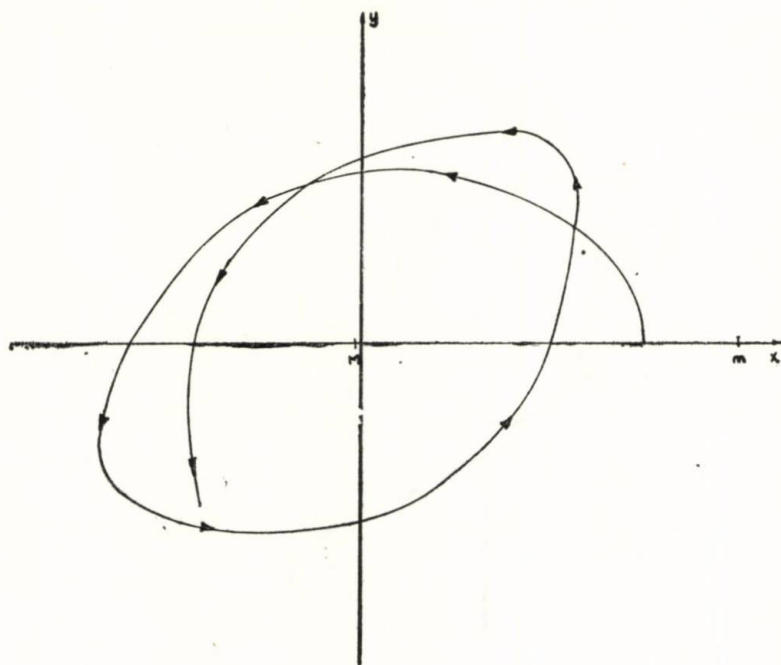


Fig. 6 $x_0 = 0,74217$ $h = -1,59406$ $\tau = 32$ jours

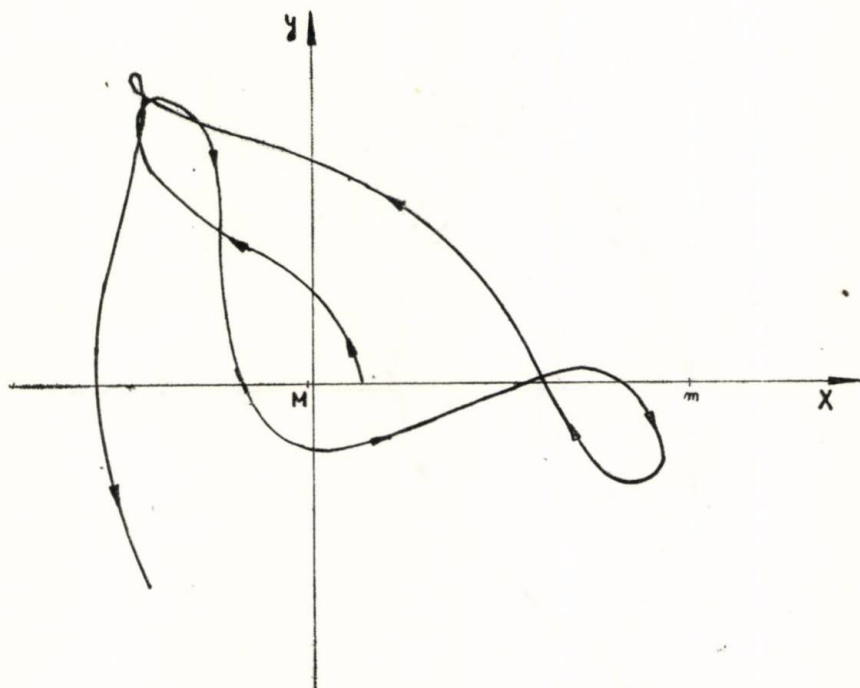


Fig. 7 $x_0 = 0,11717$ $h = -1,50606$ $\tau = 40$ jours

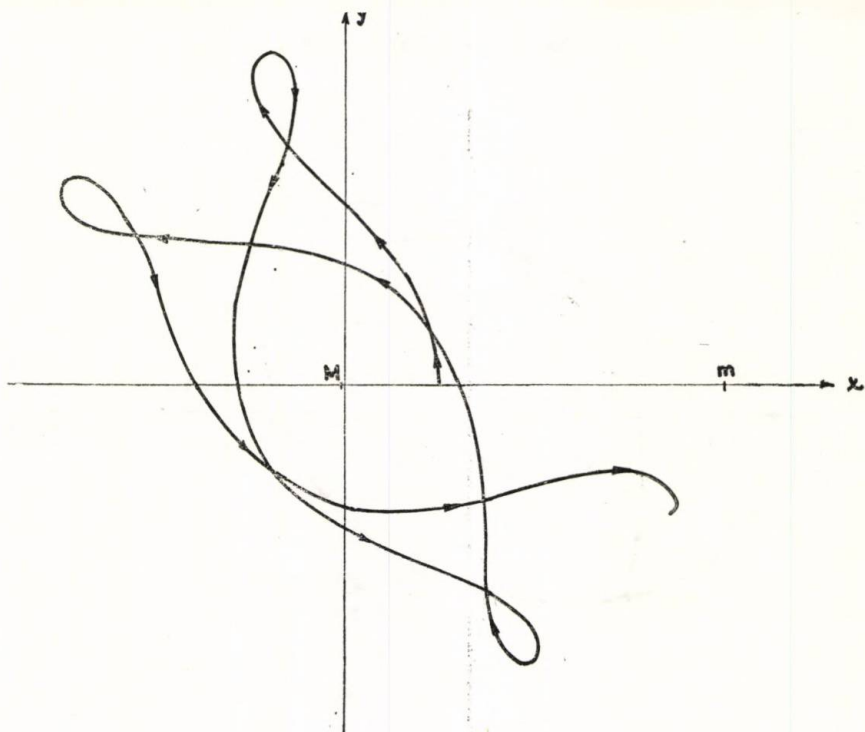


Fig. 8 $x_0 = 0,24217$ $h = -1,50606$ $\tau = 40$ jours

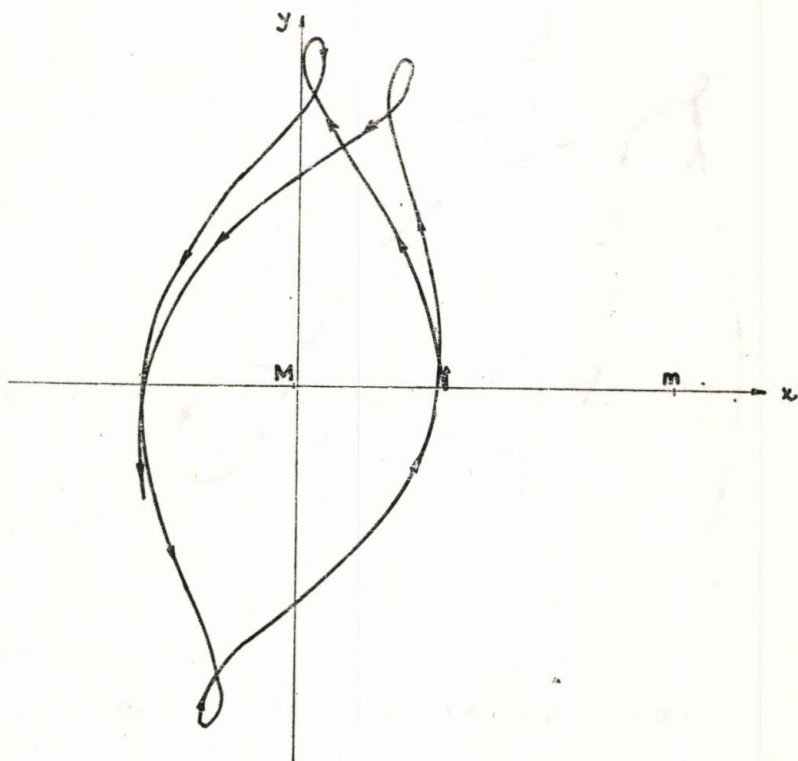


Fig. 9 $x_0 = 0,36717$ $h = -1,50606$ $\tau = 40$ jours

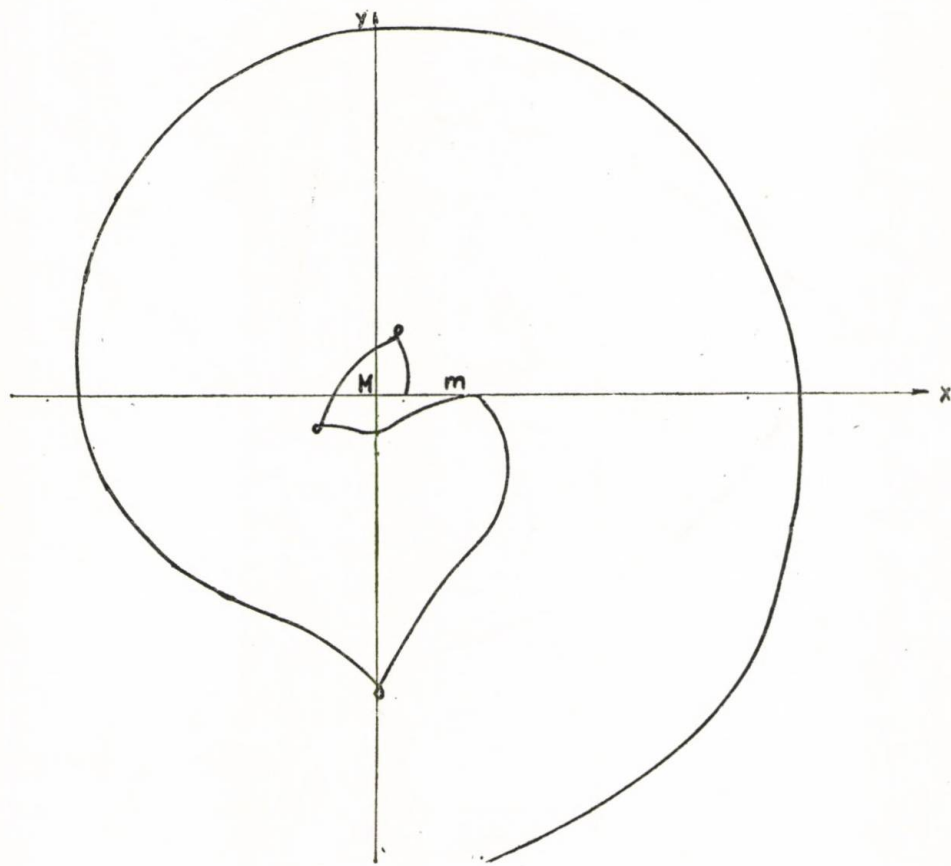


Fig. 10 $x_0 = 0,49217$ $h = -1,50606$ $\tau = 80$ jours

Les trajectoires de la masse nulle qui au moment initial t_0 partent orthogonalement à la droite Terre-Lune à des distances plus petites que 66.000 km. par rapport à la Lune sont des trajectoires d'un satellite artificiel de la Lune.

La trajectoire correspondant à $h_2 = -1,50606$ et à $x_0^4 = 0,49217$ et dont la masse nulle part au moment initial t_0 orthogonalement à la direction Terre-Lune comporte des dimensions plus grandes et contient à l'intérieur les deux masses M (Terre) et m (Lune), fig. 10.

Dans le domaine $h_1 = v - 1,59406$ on remarque deux trajectoires de la masse nulles (fig. 4, fig. 5) qui reviennent dans le voisinage du point de départ et qui, dans l'approximation du calcul numérique, peuvent conduire à des trajectoires quasi-périodiques.

Pour une petite variation de la vitesse initiale, on peut obtenir des trajectoires semblables à celles des figures 4 et 5.

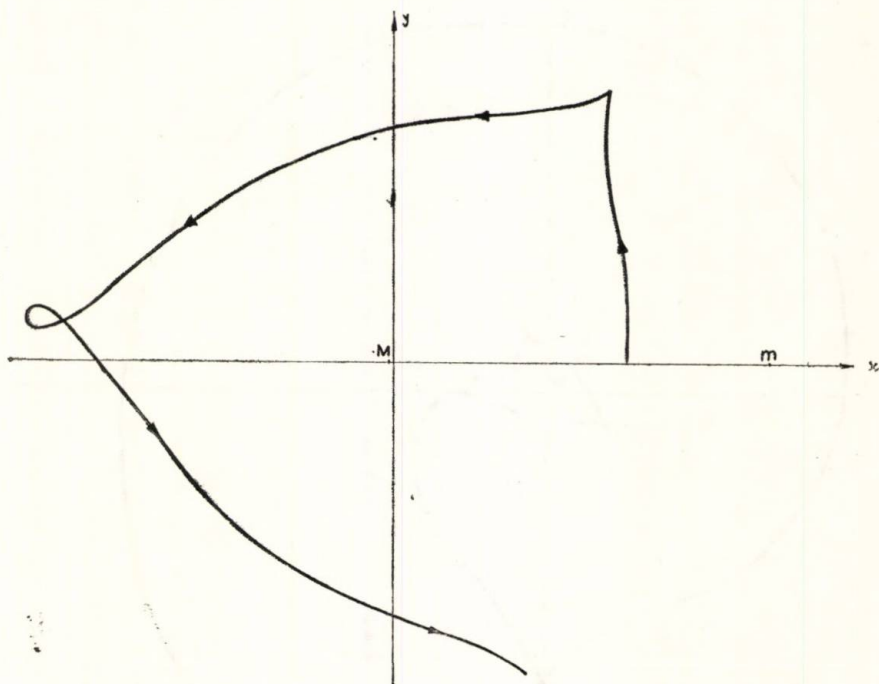


Fig. 11 $x_0 = 0,61717$ $h = -1,50606$ $\tau = 40$ jours

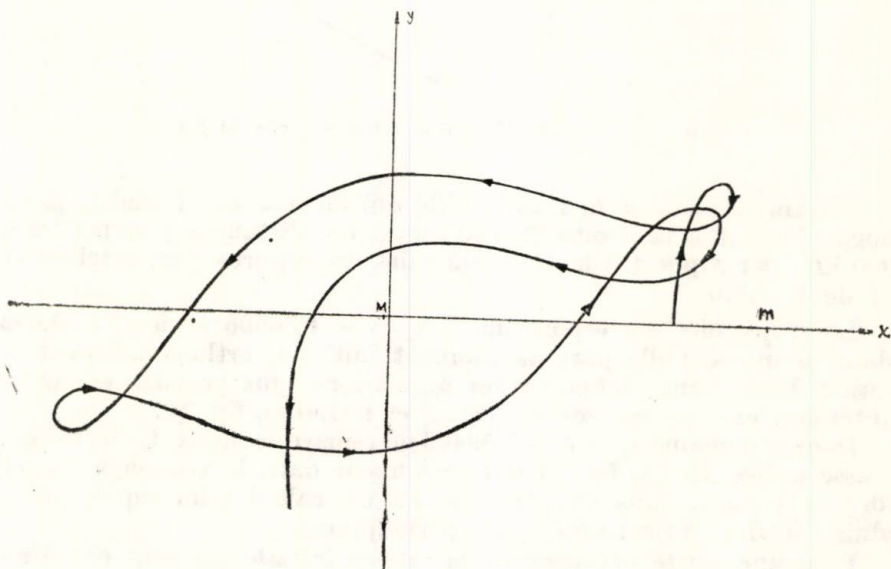


Fig. 12 $x_0 = 0,74217$ $h = -1,50606$ $\tau = 40$ jours

Pour ces trajectoires, les différences :

$$\left\{ \begin{array}{l} \Delta x = x(t_0 + T) - x(t_0) \\ \Delta y = y(t_0 + T) - y(t_0) \\ \Delta \dot{x} = \dot{x}(t_0 + T) - \dot{x}(t_0) \\ \Delta \dot{y} = \dot{y}(t_0 + T) - \dot{y}(t_0) \end{array} \right. \quad (4)$$

et les intervalles de temps T correspondant du point initial sont données dans le tableau suivant :

Position initiale	Vitesse initiale	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	T
$x_0 = 0,49217$ $y_0 = 0$	$\dot{x}_0 = 0$ $\dot{y}_0 = 1,00981$	-0,00953	-0,00059	0,04965	0,03254	24 ^j 13 ^h 05 ^m 15 ^s
	$\dot{x}_0 = 0,00533$ $\dot{y}_0 = 0,01754$	0,00167	0,00005	0,01134	-0,00562	25 ^j 14 ^h 06 ^m 34 ^s
	$\dot{x}_0 = 0$ $\dot{y}_0 = 0,01754$	0,00303	0,00026	-0,02086	-0,01017	25 ^j 14 ^h 25 ^m 25 ^s
$x_0 = 0,61717$ $y_0 = 0$	$\dot{x}_0 = 0$ $\dot{y}_0 = 0,63042$	0,03019	-0,00511	-0,11651	-0,07707	25 ^j 09 ^h 56 ^m 02 ^s
	$\dot{x}_0 = 0,00533$ $\dot{y}_0 = 0,64273$	-0,02103	0,00408	0,12375	0,04708	27 ^j 00 ^h 02 ^m 42 ^s
	$\dot{x}_0 = 0$ $\dot{y}_0 = 0,64273$	-0,01923	0,00340	0,12464	0,04267	27 ^j 09 ^h 19 ^m 30 ^s

On voit que par un choix convenable de la vitesse initiale, les courbes correspondantes aux figures 4 et 5 pourraient conduire, dans l'approximation du calcul numérique bien entendu, à des trajectoires quasi-périodiques.

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CONSTRUCTION D'UNE EXTENSION L/K DE RAMIFICATION TRIVIALE

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Dans ce mémoire on essaie de construire un corps L , extension totalement ramifiée du corps local de basse K donné sans employer l'équation d'Eisenstein. Ici on emploie une équation de forme d'Artin-Schreier généralisé. Et les résultats sont limités aux cas où le nombre de ramification est premier à la caractéristique du corps résiduel.

1. Notations

Ici on emploie les notations utilisées par Jean Pierre Serre dans le livre „Corps locaux”.

— K désigne un corps muni d'une valuation discrète v_K pour laquelle il est complet.

— A_K est l'anneau de valuation correspondant.

— m_K est son idéal maximal.

— $\bar{K} = A_K/m_K$ s'appelle le corps résiduel de K .

— v_L est la valuation discrète de l'extension L/K .

— $\deg f$ est le degré du polynôme $f(X)$.

— $U_K = A_K - m_K$, $U_K^i = 1 + m_K^i$

— $e_{L/K}$ et $f_{L/K}$ sont respectivement l'indice et le degré de ramification de l'extension L/K .

2. Proposition

Soit $f(X) \in A_K[X]$, unitaire et $f(0) = 0$, $\lambda \in K$ tel que $v_K(\lambda) = -n < 0$ et $(n, \deg f) = 1$.

On $f(X) - \lambda$ irréductible sur K et $K[X]/(f(X) - \lambda)$ est une extension totalement ramifiée de K .

Démonstration

On $f(X) = X^m + g(X)$ avec $m = \deg f$

Si $f(x) - \lambda = x^m + g(x) - \lambda = 0$ on a $v_L(x) < 0$ avec $L = K(x)$. Ce résultat est déduit de l'hypothèse $v_K(\lambda) < 0$. Car $v_L(g(x)) < v_L(x^m)$, on a $mv_L(x) = \frac{[L:K]}{f_{L/K}}(-n)$. Comme $(m, n) = 1$ on a $m = [L:K]$, cela veut dire que $f(X) - \lambda$ est irréductible.

Car $f_{L/K} | m$ et $f_{L/K} | n$, on a $f_{L/K} = 1$. C.Q.F.D.

Et en plus $v_L(x) = -n$.

3. Corollaire

a) $X^{p^m} + a_1 X^{p^{m-1}} + \dots + a_{m-1} X^p + X - \lambda = 0$ avec $a_i \in A_K$ et $(p, v_K(\lambda)) = 1$ est une équation irréductible.

b) Si $\bar{K}$ contient toutes les racines de l'équation $X^{p^m} + \bar{a}_1 X^{p^{m-1}} + \dots + \bar{a}_{m-1} X^p \pm X = 0$ et $n \leq \frac{e \cdot p^m}{p^m - 1}$ ($m > 1$) (e — l'indice de ramification absolu de K).

Alors l'extension $K(x)/K$ est galoisienne.

Démonstration

a) C'est une simple conséquence de la proposition.

b) On emploie le lemme de Hensel.

Soit $x^{p^m} + a_1 x^{p^{m-1}} + \dots + a_{m-1} x^p \pm x - \lambda = 0$, on cherche les racines de l'équation $0 = (x + Y)^{p^m} + a_1 (x + Y)^{p^{m-1}} + \dots + a_{m-1} (x + Y)^p \pm (x + Y) - \lambda \in K(x) Y$

(1) $0 = x^{p^m} + Y^{p^m} + F_0(x, Y) + a_1 x^{p^{m-1}} + a_1 Y^{p^{m-1}} + a_1 F_1(x, Y) + \dots + a_{m-1} x^p + a_{m-1} Y^p + a_{m-1} F_{m-1}(x, Y) \pm x \pm Y - \lambda$.

Là v_L — valeur minimale des coefficients de $F_0(x, Y)$ est plus grande que $v_L(p \cdot x^{p^{m-1}}) = p^m \cdot e + (p^m - 1)(-n) \geq 0$ car $n \leq \frac{e \cdot p^m}{p^m - 1}$

(chaque coefficient de $F_0(x, Y)$ a la forme $C_{p^m}^k x^k$).

En même temps la v_L — valeur minimale des coefficients de $F_{m-i}(x, Y)$ est plus grande ou égale à $v_L(p \cdot x^{p^{i-1}}) > 0$.

De là l'équation résiduelle est $H(Y) = Y + \bar{a}_i Y \pm Y = 0$ Cette équation est séparable car $H'(Y) = \pm 1 \neq 0$.

Par hypothèse et par le lemme de Hensel, le corps $K(x)$ contient toutes les racines de l'équation donnée et $K(x)$ est une extension galoisienne.

Remarques : Si $\text{Char. } K = p$ la condition de n n'est plus nécessaire.

— Les racines de l'équation (1) sont des unités dans $A_{K(x)}$.

4. Proposition

Soit $f(X) - \lambda = 0$, l'équation satisfaisant les conditions du corollaire. Si t est le plus grand entier tel que $G = G_t$, on a $t = n$ et $G_n \oplus G_{n+1} = \{1\}$

Démonstration

a) Soit $s \in G_t - G_{t+1}$ et π une uniformisante de A_L .

Alors il existe y dans A_L et qui est unité, tel que $s(x) = x + y$.

Posons $x = \pi^{-n} \cdot u$ ($u \in U_L$). Nous avons $s(\pi)/\pi = 1 + z$ avec $v_L(z) =$

$= t$ et $s(u)/u = 1 + \alpha \pi^{t+1}$ (α dans A_L). De là $s(x)/x = \frac{1}{(1+z)^n} \cdot (s(u)/u) =$

$$= \frac{1 \cdot (s(u)/u)}{1 + nz + \sum_2^n C_n^k z^k} \text{ Car } (n, p) = 1 \text{ on a } 1 + nz + \sum_2^n C_n^k z^k \text{ dans } U_L^t \text{ et}$$

ensuite $s(x)/x$ dans $U_L^t - U_L^{t+1}$. D'autre part $s(x) = x + y$ ou $s(x)/x = 1 + y \cdot x^{-1}$ dans $U_L^n - U_L^{n+1}$. Ainsi $t = n$.

b) Pour démontrer que $G = G_n \mp G_{n+1} = \{1\}$, il suffit d'avoir $v_L(s(\pi) - \pi) \leq n + 1$ avec tout s dans $G - \{1\}$.

Nous avons $x = u\pi^{-n}$ avec u dans U_L . Mais avec une simple application du lemme de Hensel, nous avons $x = a \cdot \pi'^{-n}$ (π' une autre uniformi-

sante de A_L ou $\pi' = \sqrt[n]{\frac{a}{x}}$. Nous avons $s(\pi') - \pi' = \sqrt[n]{\frac{a}{s(x)}} - \sqrt[n]{\frac{a}{x}} =$

$$= \sqrt[n]{\frac{a}{x+y}} - \sqrt[n]{\frac{a}{x}} = \frac{a/(x+y) - a/x}{\left(\sqrt[n]{\frac{a}{x+y}}\right)^{n-1} + \left(\frac{a}{x+y}\right)^{\frac{n-2}{n}} \left(\frac{a}{x}\right)^{\frac{1}{n}} + \dots + \left(\frac{a}{x}\right)^{\frac{n-1}{n}}} = \frac{A}{B}.$$

$$v_L(A) = 2n \text{ et } v_L(B) \geq v_L\left(\left(\frac{a}{x+y}\right)^{\frac{n-i}{n}} \cdot \left(\frac{a}{x}\right)^{\frac{i}{n}}\right) = \frac{n-i}{n}n +$$

$$+ \frac{i-1}{n}n = n-1$$

Done $v_L(s(\pi') - \pi') \leq 2n - (n-1) = n-1$.

5. Corollaire

a) G est un groupe p -élémentaire.

b) G est isomorphe au groupe additif des racines de l'équation $0 = Y^p + \bar{a}_1 Y^{p-1} + \dots + \bar{a}_{m-1} Y + \bar{a}_m \pm Y \in \bar{K}[Y]$.

Démonstration

a) est une conséquence directe de la proposition.

Car $s(y_i) - y_i \in m_L^{n+1}$ avec y_i les racines de l'équation (1) ($1 \leq i \leq \leq p^m$ et tous s dans G). Soit $s(x) - x = y_i$ et $t(x) - x = y_k$, on a $ts(x) - x = x + y_k + y_i + b\pi^{n+1}$

Considérons l'application $f: G \rightarrow \bar{K}$ par $f(s) = \bar{y}_i$.

Alors $f(ts) = \bar{y}_k + \bar{y}_i$. Et si $f(s) = 0$, on a $s(x) - x = 0$ ou $s = id$. car $L = \bar{K}(x)$. De là f est injective. En plus f est surjective car $(G:1) = p^m$

En combinant la proposition d'en haut avec le résultat de Hase-winkel on trouve une méthode de construire une extension galoisienne totalement ramifiée de ramification triviale de tout nombre n donné premier à p avec tout groupe G p -élémentaire donné, en choisissant le corps de base K convenable.

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ON COMPUTER GENERATION OF BURR AND PARETO RANDOM VARIABLES

ILEANA POPESCU

1. Introduction

The aim of this paper is to give several algorithms for generating Burr and Pareto random variables. Some algorithms for generating multivariate Burr and Pareto random variables and conditional variables obtained from these are also presented.

Each algorithm has a theoretical support explaining its elaboration and generates a single sample value. FORTRAN programs were written and validated on an IBM 360/30 at the Computer Centre of the University of Bucharest.

2. Generation of Burr random variable

A random variable X has a Burr distribution $Br(c, k)$ if its distribution function takes on the following form :

$$F(x) = \begin{cases} 1 - \frac{1}{(1+x^c)^k} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where $c, k \in R_+$. The mean value and variance are :

$$E(X) = kB \left(k - \frac{1}{c}, 1 + \frac{1}{c} \right) \quad (2)$$

and

$$D^2(X) = kB \left(k - \frac{2}{c}, 1 + \frac{2}{c} \right) - k^2 B^2 \left(k - \frac{1}{c}, 1 + \frac{1}{c} \right) \quad (3)$$

where $B(p, q)$ is the beta function.

In the sequel we shall give some algorithms for computer generating of this variable.

The inverse transformation method is based on Hincin's lemma [3] and is applied to those distribution functions F for which F^{-1} can be easily calculated. In this case sample values for X are obtained from $X = F^{-1}(U)$, where U is an uniform random variable over $[0, 1]$.

Hence, in our case :

$$X = [(1 - U)^{-\frac{1}{k}} - 1]^{\frac{1}{c}}$$

Using the fact that $1 - U$ is also uniformly distributed over $[0, 1]$ we have the following algorithm.

Algorithm 1

0. Input (c, k) .

1. Generate U (subroutine RANDU from S.S.P.).

2. Calculate $X = [(1 - U)^{-\frac{1}{k}} - 1]^{\frac{1}{c}}$.

3. Deliver X .

Proposition 1. Burr distribution is a mixture of a Weibull distribution $W(\theta, c)$, θ having a gamma distribution $\Gamma(k, 1)$.

Proof.

Let $G(x) = 1 - e^{-\theta x^c}$, $\theta > 0$, $x > 0$, be the distribution function of a Weibull random variable $W(\theta, c)$ and

$$f(t) = \frac{t^{k-1} e^{-t}}{\Gamma(k)}, \quad t > 0, \quad k > 0$$

the probability density function of a gamma random variable $\Gamma(k, 1)$
Hence

$$\begin{aligned} \frac{1}{\Gamma(k)} \int_0^\infty (1 - e^{-\theta x^c}) \theta^{k-1} e^{-\theta} d\theta &= \frac{1}{\Gamma(k)} \int_0^\infty e^{-\theta} \theta^{k-1} d\theta - \\ &- \frac{1}{\Gamma(k)} \int_0^\infty e^{-\theta(1+x^c)} \theta^{k-1} d\theta = \\ &= 1 - \frac{1}{\Gamma(k) (1+x^c)^{k-1}} \int_0^\infty e^{-\theta(1+x^c)} [\theta(1+x^c)]^{k-1} d\theta = \\ &= 1 - \frac{1}{\Gamma(k) (1+x^c)^k} \int_0^\infty e^{-t} t^{k-1} dt = 1 - \frac{1}{(1+x^c)^k}. \end{aligned}$$

This proposition allows us to use the composition method [1].

Algorithm 2

0. Input (c, k) .

1. Generate θ , a random variable $\Gamma(k, 1)$.

2. Generate X , a random variable $W(\theta, c)$.

3. Deliver X .

Proposition 2. If Y is a Pareto distributed $P(a = k, k = 1)$ random variable, then $X = (Y - 1)^{\frac{1}{c}}$ is Burr distributed $Br(c, k)$.

Proof.

$$F(x) = P(X < x) = P(Y < 1 + x^c) = \int_k^{1+x^c} \frac{k}{t^{k+1}} dt$$

Hence

$\frac{d}{dx} F(x) = \frac{k c x^{c-1}}{(1+x^c)^{k+1}}$ which is the probability density function of a Burr variable $Br(k, c)$.

Algorithm 3

0. Input (c, k) .
1. Generate Y , a random variable $P(a = k, k = 1)$ (see algorithms 7–11)
2. Calculate $X = (Y - 1)^{1/c}$
3. Deliver X

Takahasi [6] has shown that :

$$\begin{aligned} & \int_0^\infty \prod_{j=1}^m c_j \rho_j \theta x_j^{c_j-1} e^{-\rho_j \theta x_j^{c_j}} \frac{\alpha^k e^{-\alpha \theta} \theta^{k-1}}{\Gamma(k)} d\theta = \\ & = \frac{\alpha^k \Gamma(k+m)}{\Gamma(k)} \prod_{j=1}^m c_j \rho_j \frac{\prod_{j=1}^m x_j^{c_j-1}}{\left(\alpha + \sum_{j=1}^m \rho_j x_j^{c_j} \right)^{k+m}} \end{aligned} \quad (4)$$

Let us denote $\frac{\rho_j}{\alpha} = \alpha_j$ and define

$$f(x_1 \dots x_m; c_1 \dots c_m; \alpha_1 \dots \alpha_m; k) = \begin{cases} \frac{\Gamma(k+m)}{\Gamma(k)} \prod_{j=1}^m c_j \alpha_j x_j^{c_j-1} \frac{1}{\left(1 + \sum_{j=1}^m \alpha_j x_j^{c_j} \right)^{k+m}} & \text{if } x_j > 0, 1 \leq j \leq m \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where $c_j \geq 1$, $\rho_j > 0$, $\alpha_j > 0$, $j = 1, \dots, m$ and $\theta > 0$.

A random variable $X = (X_1, \dots, X_m)$ with probability density function given in (5) is called a m -variate Burr random variable.

From (4) we see that this random variable is a mixture of a m -variate Weibull variable with a gamma variable. Now it is easily seen that each $X_j, j = 1, \dots, m$ is Burr distributed. In fact, $(\alpha_j)^{\frac{1}{c_j}} X_j$ is Burr distributed with $c = c_j$.

Algorithm 4

0. Input $(c_j, \rho_j, k, \alpha, m)$.

1. Generate a gamma random variable $\Gamma(k, \alpha)$ which is the parameter θ of a m -variate Weibull distribution.

2. Generate a m -variate Weibull random variable $X = (X_1, \dots, X_m)'$ with parameters θ, c_j, ρ_j .

Taking into account that $(Z)^{1/c}$ is Weibull distributed if Z is exponentially distributed, the generation of a m -variate Weibull random variable requires only the calculation of $X_i = (Z_i)^{1/c_i}$ where $Z = (Z_1, \dots, Z_m)'$ is a m -variate exponentially distributed random variable.

3. Deliver X (a sample value of a m -variate Burr random variable with parameters $c_j, \frac{\rho_j}{\alpha} = \alpha_j, k$).

Hincin's lemma can be generalized for random vectors, but the system of equations obtained is sometimes hardly to solve.

For instance, in our case, for $m = 2$ the distribution function is

$$F(x_1, x_2) = 1 - \frac{1}{1 + \alpha_1 x_1^{c_1}} - \frac{1}{1 + \alpha_2 x_2^{c_2}} + \frac{1}{1 + \alpha_1 x_1^{c_1} + \alpha_2 x_2^{c_2}}$$

Hence,

$$F_1(x_1) = P(X_1 < x_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{+\infty} dF(x_1, x_2) = 1 - \frac{1}{1 + \alpha_1 x_1^{c_1}}$$

$$\begin{aligned} F_2(x_1, x_2) &= P(X_2 < x_2 | X_1 = x_1) = \int_{-\infty}^{x_2} \frac{f(x_1, x_2)}{\int_{-\infty}^{+\infty} f(x_1, x_2) dx_2} dx_2 = \\ &= \int_0^{x_2} \frac{2\alpha_2 x_2^{c_2-1} (1 + \alpha_1 x_1^{c_1})^2}{(1 + \alpha_1 x_1^{c_1} + \alpha_2 x_2^{c_2})^3} dx_2 = 1 - \frac{(1 + \alpha_1 x_1^{c_1})^2}{(1 + \alpha_1 x_1^{c_1} + \alpha_2 x_2^{c_2})^2} \end{aligned}$$

and let be the system:

$$F_1(X_1) = U_1 \quad (6)$$

$$F_2(X_1, X_2) = U_2$$

where U_1 and U_2 are independent and uniformly distributed over $[0, 1]$. The solution of (6) is

$$X_1 = \left[\frac{U_1}{\alpha_1 (1 - U_1)} \right]^{\frac{1}{c_1}} \quad (7)$$

$$X_2 = \left[\frac{1 - (1 - U_2)^{\frac{1}{c_2}}}{\alpha_2 (1 - U_1) (1 - U_2)^{\frac{1}{c_2}}} \right]^{\frac{1}{c_2}}$$

and the joined distribution of X_1, X_2 is $F(x_1, x_2)$.

Algorithm 5

0. Input $(\alpha_1, \alpha_2, c_1, c_2)$
 1. Generate U_1, U_2 independent and uniformly distributed over $[0, 1]$ random variables (subroutine RANDU from S.S.P.).
 2. Calculate $F_1(x_1)$ and $F_2(x_1, x_2)$
 3. Solve the system (6)
 4. Deliver $X = (X_1, X_2)$.
- The marginal and conditional distribution derived from this distribution are also Burr distributed with modified parameters.

Thus (see [6]):

$$f(x_1, \dots, x_r | X_{r+1} = \eta_{r+1} \dots X_m = \eta_m; c_1, \dots, c_m; \alpha_1 \dots \alpha_m; k) = \\ = f\left(x_1, \dots, x_r; c_1, \dots, c_r; \frac{\alpha_1}{1 + \sum_{j=r+1}^m \alpha_j \eta_j^{c_j}}, \dots, \frac{\alpha_r}{1 + \sum_{j=r+1}^m \alpha_j \eta_j^{c_j}}; k+m-r\right)$$

Hence, for instance, the conditional distribution of $\left(\frac{\alpha_1}{1 + \sum_{j=2}^m \alpha_j \eta_j^{c_j}}\right)^{\frac{1}{c_1}} X_1$

given $X_2 = \eta_2 \dots X_m = \eta_m$ is of the form (1) with c replaced by c_1 and k replaced by $k + m - 1$.

If $X = (X_1, \dots, X_m)$ is Burr distributed and if the components of X are rearranged so that the first r components are unknown and the others $m - r$ are known then the problem is to generate $X^{(1)} = (X_1, \dots, X_r)'$ given $X^{(2)} = (X_{r+1}, \dots, X_m)' = (\eta_{r+1}, \dots, \eta_m)'$. Considering the procedure of generating of the conditional data for multivariate normal distribution [7] we have:

Algorithm 6

0. Input (c_j, α_j, m, r, k) .
1. Specify elements and parameters of $X^{(2)}$.
2. Calculate $\alpha_i^* = \frac{\alpha_i}{1 + \sum_{j=r+1}^m \alpha_j \eta_j^{c_j}}, 1 \leq i \leq r$.
4. Generate a r -variate Burr distributed random variable X with parameters $c_j, \alpha_j^*, k + m - r, 1 \leq j \leq r$.
5. Deliver X .

3. Generation of Pareto random variable

A random variable X has a Pareto distribution $P(a, k)$ if its distribution function takes on the following form :

$$F(x) = 1 - \left(\frac{k}{x}\right)^a, \quad x \geq k \quad (8)$$

where $k > 0$, $a > 0$. The mean variance are :

$$E(X) = \frac{ak}{a-1} \quad \text{if } a > 1 \quad (9)$$

and

$$D^2(X) = \frac{ak^2}{(a-1)(a-2)} \quad \text{if } a > 2 \quad (10)$$

Taking into account some of the theoretical considerations for generating Burr variables and the properties of Pareto variables we get some procedures for generating the latter ones.

The inverse transformation method leads to :

Algorithm 7

0. Input (k, a) .

1. Generate U , a uniformly distributed over $[0, 1]$ random variable

2. Calculate $X = k U^{-\frac{1}{a}}$ (applying Hincin's lemma we have

$$X = \frac{k}{(1 - U)^{1/a}})$$

3. Deliver X .

Harris [4] obtained a special form of Pareto's distribution called the Lomax distributin as a mixture of exponential distributions $G(x) = 1 - e^{-\frac{x}{\theta}}$, θ^{-1} having a gamma distribution whose density functions is

$$h(x) = \frac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha} \Gamma(\alpha)}$$

In order to obtain (8) we consider a mixture of

$$G(x) = 1 - e^{-\theta(x-k)}, \quad x \geq k \quad (11)$$

θ having the density function

$$h(x) = \frac{x^{a-1} e^{-xk} k^a}{\Gamma(a)} \quad (12)$$

Indeed :

$$\int_0^\infty \frac{t^{a-1} e^{-tk} k^a}{\Gamma(a)} (1 - e^{-t(x-k)}) dt = 1 - \int_0^\infty \frac{t^{a-1} e^{-tx}}{\Gamma(a)} k^a dt = 1 - \left(\frac{k}{x}\right)^a$$

Applying the composition method we have :

Algorithm 8

0. Input (k, a) .

1. Generate Y , a random variable with density (12).

2. Generate X , a random variable with density (11) where $\theta = Y$.

3. Deliver X .

Proposition 3. If Y is a random variable with density $f(y) = e^{-y}$, $y > 0$, then the random variable $X = k e^{y/a}$ is a Pareto random variable.

Proof.

$$P(X) < z) = P(k e^{y/a} < z) = P\left(Y < a \ln \frac{z}{k}\right) = \int_0^{a \ln \frac{z}{k}} e^{-t} dt$$

Hence

$$\frac{d}{dz} P(X < z) = \frac{ak^a}{z^{a+1}} \text{ which is the density function corresponding to (8).}$$

This proposition leads to an algorithm similar to the algorithm 7, if we generate Y using the inverse transformation method.

Proposition 4. If Y is a random variable with density function $g(y) = ak^a y^{a-1}$, $0 \leq y \leq \frac{1}{k}$ (the power distribution function) then $X = \frac{1}{Y}$ is a Pareto random variable $P(a, k)$.

Proof.

$$P(X < x) = P\left(\frac{1}{Y} < x\right) = P\left(Y > \frac{1}{x}\right) = 1 - \int_0^{\frac{1}{x}} ak^a t^{a-1} dt$$

Hence,

$$\frac{d}{dx} P(X < x) = \frac{ak^a}{x^{a+1}} \text{ which is a density function corresponding to (8).}$$

A modified form of the rejection method [9] was used in order to generate the random variable Y .

Algorithm 10

0. Input (a, k, g, M) .
1. Generate U_1 and U_2 independent and uniformly distributed over $[0, 1]$ random variables.
2. Calculate $V_1 = \frac{1}{k} U_1$.
3. If $U_2 \geq \frac{g(V_1)}{M}$ reject (U_1, U_2) and go back to step 1.
4. (When the pair (U_1, U_2) is accepted) Take $Y = V_1$.
5. Calculate $X = \frac{1}{Y}$.
6. Deliver X .

Another algorithm for generating this random variable is based on its functional relation with the Champernowne random variable [2] whose distribution is :

$$G(t) = 1 - \frac{2}{\theta} \cdot \frac{1}{\operatorname{tg} \left[\frac{\sin \theta}{\cos \theta + \left(\frac{t}{t_0} \right)^2} \right]}, \quad t > 0$$

where $\alpha > 0$.

For θ close to zero $G(t)$ becomes

$$G_1(t) = \frac{\left(\frac{t}{t_0} \right)^\alpha}{1 + \left(\frac{t}{t_0} \right)^\alpha} = 1 - \frac{\left(\frac{t}{t_0} \right)^{-\alpha}}{1 + \left(\frac{t}{t_0} \right)^{-\alpha}}$$

and for t close to zero $G_1(t)$ is an approximation of the Pareto distribution.

For $\alpha \geq 1$, Bucher's technique (which is a combination of the composition and rejection methods) is applicable provided that the generated random variable has a density function of the form :

$$f(x) = \sum_{i=1}^p \alpha_i f_i(x) g_i(x)$$

where f_i , $1 \leq i \leq p$ are density functions, α_i positive constants and g_i are functions taking on values in $[0, 1]$.

In our case,

$$f(x) = \frac{ak^a}{x^{a+1}} = a \left(\frac{k}{x} \right)^{a-1} \frac{k}{x^2} \quad x \geq k > 0, \quad a \geq 1$$

$$\alpha_1 = a$$

$$f_1(x) = \frac{k}{x^2}$$

$$g_1(x) = \left(\frac{k}{x} \right)^{a-1}$$

Algorithm 11

0. Input (k, a) .
1. Generate U , a uniformly distributed random variable over $[0, 1]$ (subroutine RANDU from S.S.P).
2. Generate a sample value Y corresponding to $f_1(y) \cdot \left(Y = \frac{k}{1 - 2U} \right)$.
3. Calculate $g_1(Y)$.
4. If $U > g_1(Y)$ reject Y and go back to step 2.
5. (When Y is accepted) take $X = Y$ which is a sample value corresponding to $f(x)$.
6. Deliver X .

There are two types of multivariate Pareto distributions. One in which $a_i = a$ for $1 \leq i \leq n$ and the other for which this conditions is not necessary. In the first case, the density function is given by

$$f(x_1, \dots, x_m) = a(a+1), \dots, (a+m-1)$$

$$\left(\prod_{i=1}^m k_i \right)^{-1} \left(\sum_{i=1}^m \frac{x_i}{k_i} - m + 1 \right), \quad x_i > k_i > 0$$

The derived marginal and conditional distributions are of the same type with parameters a and k modified correspondingly.

Let us generate sample values of $X^{(2)} = (X_{r+1}, \dots, X_m)'$ given that $X_1 = \eta_1, \dots, X_r = \eta_r$.

Algorithm 12

0. Input $(a, k_i, 1 \leq i \leq m, m, r)$
1. Specify the elements and parameters of $X^{(1)} = (X_1, \dots, X_r)'$
1. $k_i^* = k_i \left(\sum_{l=1}^r k_l^{-1} \eta_l - r + 1 \right)$ for $r+1 \leq i \leq m$.

2. Generate X , $a(m-r)$ -variate Pareto variable with parameters

$$a \rightarrow a + r, k_i \rightarrow k_i^*.$$

3. Deliver X .

The second type m -variate Pareto distribution was introduced by Mardia [5].

If $X_1, \dots, X_n$ are all Pareto distributed with $f_{X_i}(x_i) = a_i k_i^{a_i} (x_i)^{-(a_i+1)}$ then the variables

$$Y_i = a_i \log \left(\frac{X_i}{k_i} \right); 1 \leq i \leq n, x_i > k_i > 0, a_i > 0 \quad (13)$$

have multivariate exponential distribution of a certain form, for which there is a corresponding multivariate Pareto distribution.

Different types of multivariate Pareto distribution may be obtained from different multivariate exponential distributions applying transformation (13) and conversely.

Practical considerations

Let $X_1, \dots, X_N$ be a N -sized sample on the variable X . The validation criterion consisted in the comparison of the theoretical mean $E(X)$ with the sample mean given by:

$$\bar{X} = \frac{1}{N} \sum_{i=1}^N X_i$$

and of the theoretical variance $D^2(X)$ with the sample variance

$$S^2 = \frac{1}{N-1} \sum_{i=1}^N (X_i - \bar{X})^2$$

In computation we considered $N = 100$, and the time mentioned includes also the calculation of $\bar{X}$ and S^2 . For Burr variable we supposed that $c = 10$, $k = 10$ and for Pareto variable we have taken $k = 6$, $a = 8$. The obtained results are given in tables 1 and 2.

For computer generating of Burr variable is to be seen that the algorithms 1 and 3 are the best, but the first one is preferable if we take into account also the necessary memory space.

For the Pareto variable the best is the algorithm 9 which gives the same results as 7 (see the theoretical arguments given) but it requires a shorter time. Generally, all the algorithms for generating the Pareto variable give quite good results. Even if the algorithm 10, for instance, requires a longer period of time for generating 100 Pareto sample values, it gives a better accuracy for $\bar{X}$ and S^2 .

Table 1 (generation of Burr variable)

Algorithm	Time	Theoretical mean	Sample mean	Theoretical mean	Sample variance
1	1'36"	0,8	0,758836	0,006	0,0071045
2	2'07"	0,8	0,69721	0,006	0,005069
2	1'39"	0,8	0,727663	0,006	0,006998
modif. 3	1'36"	0,8	0,758836	0,006	0,0071045

Table 2 (generation of Pareto variable)

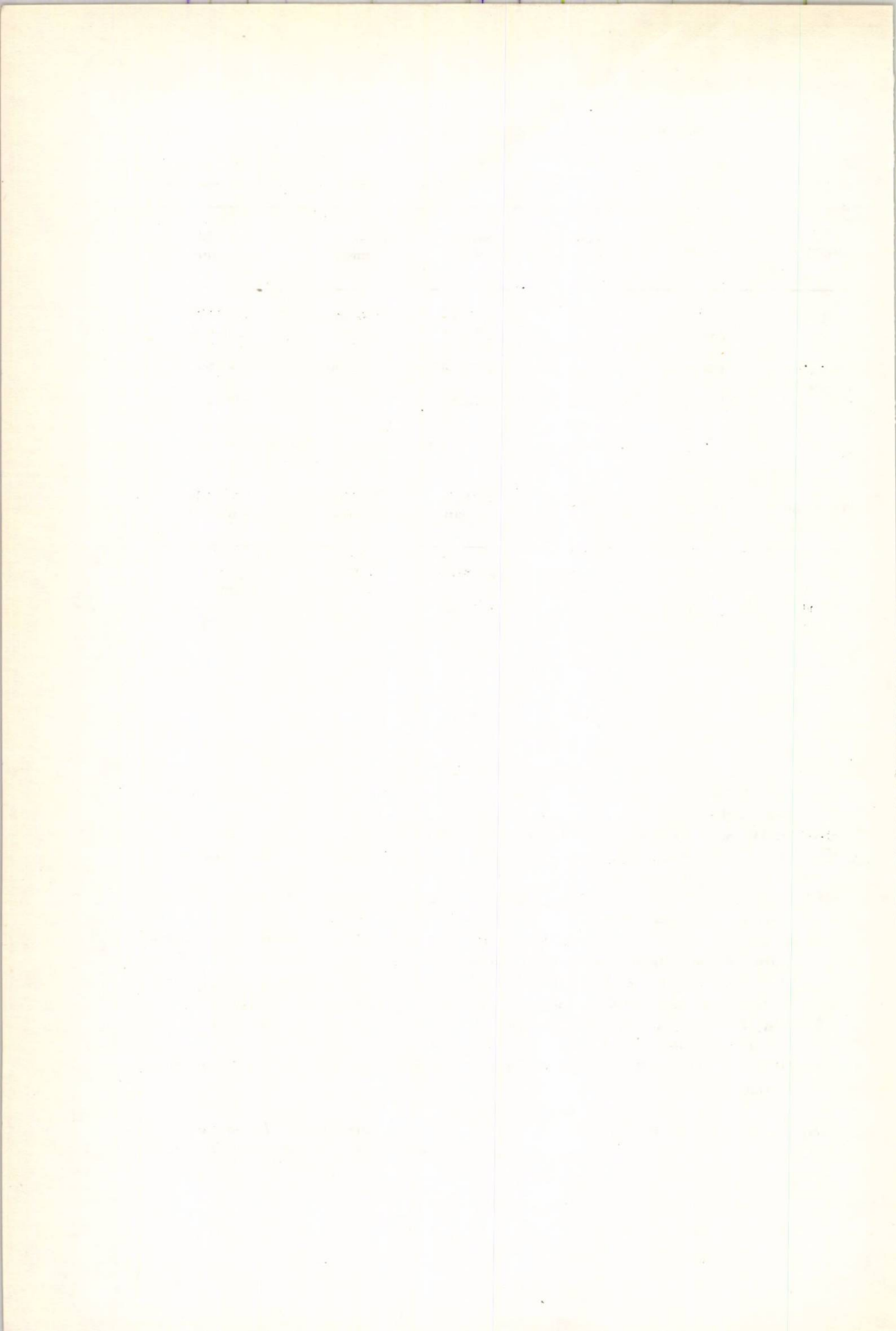
Algorithm	Time	Theoretical mean	Sample mean	Theoretical mean	Sample variance
7	1'35"	6,8571	6,77533	6,8571	6,29143
8	1'36"	6,8571	6,72545	6,8571	6,380161
9	1'31"	6,8571	6,77533	6,8571	6,29143
10	1'53"	6,8571	7,00181	6,8571	6,65624
11	1'43"	6,8571	7,06881	6,8571	7,59432

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DEFINITE INTEGRALS INVOLVING MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

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§ Introduction. Little is known about definite integrals for modified Bessel functions of the second kind. The object of this paper is to establish such results and the following formulae are established

$$\begin{aligned} & \int_0^Z y^{\frac{nu}{2} - \gamma - 1} K_{2\nu} \left(2^{\frac{3}{2}} y^{n/4} e^{\frac{1}{4} \pi i} \right) K_{2\nu} \left(2^{\frac{3}{2}} y^{n/4} e^{-\frac{1}{4} \pi i} \right) dy = \\ &= \frac{1}{n 2^3 \pi^{\frac{1}{2}}} \Gamma \left(\frac{1}{2} u + \nu - \frac{\gamma}{n} \right) \Gamma \left(\frac{1}{2} u - \nu - \frac{\gamma}{n} \right) \Gamma \left(\frac{1}{2} u - \right. \\ & \left. - \frac{\gamma}{n} \right) \Gamma \left(\frac{1}{2} u + \frac{1}{2} - \frac{\gamma}{n} \right) - \frac{Z^{-\gamma}}{n 2^4 \pi^{\frac{3}{2}}} - \\ & - \sum_{i, -i} \frac{1}{i} {}_6E_1 \left[\begin{matrix} \frac{1}{2} u + \nu, \frac{1}{2} u - \nu, \frac{1}{2} u, \frac{1}{2} u + \frac{1}{2}, 1, \frac{\gamma}{n} : e^{i\pi} Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right] \quad (1) \end{aligned}$$

where the symbol $\sum_{i, -i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are to be added.

$$\begin{aligned} & \int_0^Z y^{\frac{nu}{2} - \gamma - 1} K_{\nu}^2(y^{n/2}) dy = \\ &= \frac{\pi^{\frac{3}{2}} \Gamma \left(\nu + \frac{1}{2} u - \frac{\gamma}{n} \right) \Gamma \left(-\nu + \frac{1}{2} u - \frac{\gamma}{n} \right) \Gamma \left(\frac{1}{2} u - \frac{\gamma}{n} \right)}{2n \Gamma \left(\frac{1}{2} u + \frac{1}{2} - \frac{\gamma}{n} \right)} - \\ & - \frac{\pi^{\frac{1}{2}}}{4n} Z^{-\gamma} \sum_{i, -i} \frac{1}{i} {}_5E_2 \left[\begin{matrix} \nu + \frac{1}{2} u, -\nu + \frac{1}{2} u, \frac{1}{2} u, 1, \frac{\gamma}{n} : Z^n e^{i\pi} \\ \frac{1}{2} u + \frac{1}{2}, \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (2) \end{aligned}$$

$$\int_0^Z y^{\frac{nu}{2} - \gamma - 1} K_\nu(2y^{n/2}) dy = \frac{1}{2n} \Gamma\left(\frac{1}{2}u + \frac{1}{2}\nu - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2}u - \frac{1}{2}\nu - \frac{\gamma}{n}\right) -$$

$$- \frac{Z^{-\nu}}{4n\pi} \sum_{i, -i} \frac{1}{i} {}_4E_1 \left[\begin{matrix} \frac{1}{2}u + \frac{1}{2}\nu, \frac{1}{2}u - \frac{1}{2}\nu, 1, \frac{\gamma}{n} : e^{i\pi} Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (3)$$

$$\int_0^Z y^{an - \gamma - 1} K_{2(b-a)}(2^{\frac{3}{2}} y^{n/4} e^{\frac{1}{4}\pi i}) K_{2(a-b)}(2^{\frac{3}{2}} y^{n/4} e^{-\frac{1}{4}\pi i}) dy =$$

$$= \frac{1}{n} 2^{-3} \pi^{-\frac{1}{2}} \Gamma\left(a - \frac{\gamma}{n}\right) \Gamma\left(b - \frac{\gamma}{n}\right) \Gamma\left(a + \frac{1}{2} - \frac{\gamma}{n}\right) \Gamma\left(2a - b - \frac{\gamma}{n}\right) -$$

$$- \frac{1}{n} 2^{-4} \pi^{-\frac{3}{2}} Z^{-\gamma} \sum_{i, -i} \frac{1}{i} {}_6E_1 \left[\begin{matrix} 1, a, b, a + \frac{1}{2}, 2a - b, \frac{\gamma}{n} : e^{i\pi} Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (4)$$

$$\int_0^Z y^{\frac{la}{u} + \frac{n}{2u} - \gamma - 1} e^{-\frac{1}{2}(uy^{n/2}u)} K_m\left(\frac{ny^{n/2}}{2}\right) dy =$$

$$= \frac{\pi^{\frac{1}{2}}}{n} (2\pi)^{-\frac{1}{2}(u-1)} \frac{\Gamma\Delta\left(u; \frac{1}{2} + m + l - \frac{\gamma}{n}\right) \Gamma\Delta\left(u; \frac{1}{2} - m + l - \frac{\gamma}{n}\right)}{\Gamma\Delta\left(u; 1 + l - \frac{\gamma}{n}\right) \Gamma\left(1 - \frac{\gamma}{n}\right)} -$$

$$- \frac{Z^{-\gamma}}{n} \pi^{\frac{1}{2}} (2\pi)^{-\frac{1}{2}(u+1)} \sum_{i, -i} \frac{1}{i}$$

$$E \left[\begin{matrix} \Delta\left(u; \frac{1}{2} + m + l\right), \Delta\left(u; \frac{1}{2} - m + l\right), \frac{\gamma}{n} : Z^n e^{i\pi} \\ \Delta(u; 1 + l), \frac{\gamma}{n} + 1 \end{matrix} \right], \quad (5)$$

where u is any positive integer, the symbol $\Delta(u, \alpha)$, say, represents the set of parameters

$$\frac{\alpha}{u}, \frac{\alpha+1}{u}, \dots, \frac{\alpha+u-1}{u}$$

and the symbol $\Gamma\Delta(u; \alpha)$, say,

$$= \Gamma\left(\frac{\alpha}{u}\right) \Gamma\left(\frac{\alpha+1}{u}\right) \dots \Gamma\left(\frac{\alpha+u-1}{u}\right).$$

$$\int_0^Z y^{n(l+\frac{1}{2})-\gamma-1} e^{-\frac{1}{2}y^n} K_m\left(\frac{1}{2}y^n\right) dy =$$

$$= \frac{\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2} + m + l - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2} - m + l - \frac{\gamma}{n}\right)}{n \Gamma\left(1 + l - \frac{\gamma}{n}\right)} -$$

$$- \frac{Z^{-\gamma}}{2n \pi^{\frac{1}{2}}} \sum_{i=-i}^i \frac{1}{i} {}_4E_2 \left[\begin{matrix} \frac{1}{2} + m + l, \frac{1}{2} - m + l, 1, \frac{\gamma}{n} : Z^n e^{i\pi} \\ 1 + l, \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (6)$$

$$\int_0^Z y^{\frac{lu}{u} + \frac{n}{u} - \gamma - 1} K_m(2u^2 y^{n/u}) K_m(2u^2 y^{n/u}) dy =$$

$$= \frac{\Gamma\Delta\left(u; \frac{1}{2} + \frac{1}{2}l - \frac{\gamma}{n}\right) \Gamma\Delta\left(u; \frac{1}{2}l + m + \frac{1}{2} - \frac{\gamma}{n}\right) \Gamma\Delta\left(u; \frac{1}{2}l - m - \frac{1}{2} - \frac{\gamma}{n}\right)}{n 2^{l+u+1} u^{3/2} \pi^{u-3/2} \Gamma\Delta\left(u; 1 - \frac{1}{2}l - \frac{\gamma}{n}\right) \Gamma\left(1 - \frac{\gamma}{n}\right)} -$$

$$- \frac{Z^{-\gamma}}{n 2^{l+u+2} u^{3/2} \pi^{u-1/2}} \sum_{i=-i}^i \frac{1}{i} {}_4E_2 \left[\begin{matrix} \Delta\left(u; \frac{1}{2} + \frac{1}{2}l\right), \Delta\left(u; \frac{1}{2}l + m + \frac{1}{2}\right), \Delta\left(u; \frac{1}{2}l - m + \frac{1}{2}\right), \frac{\gamma}{n} : Z^n e^{i\pi} \\ \Delta\left(u; 1 - \frac{1}{2}l\right), \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (7)$$

$$\int_0^Z y^{-\gamma-1} K_u(2y^{n/2}) dy = \frac{1}{2n} \Gamma\left(\frac{1}{2}u - \frac{\gamma}{n}\right) \Gamma\left(-\frac{1}{2}u - \frac{\gamma}{n}\right) - \frac{Z^{-\gamma}}{4n\pi} \sum_{i=-i} \frac{1}{i} {}_4E_1 \left[\begin{matrix} 1, \frac{1}{2}u, -\frac{1}{2}u, \frac{\gamma}{n} : Z^n e^{i\pi} \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (8)$$

$$\begin{aligned} \int_0^Z y^{n/2-\gamma-1} K_m(y^{n/2}) K_n(y^{n/2}) dy &= \frac{\sqrt{\pi}}{2n} \Gamma\left(\frac{1}{2} + \frac{1}{2}m + \frac{1}{2}n - \frac{\gamma}{n}\right) - \\ &- \frac{\Gamma\left(\frac{1}{2} + \frac{1}{2}m - \frac{1}{2}n - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2} + \frac{1}{2}n - \frac{1}{2}m - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m - \frac{\gamma}{n}\right)}{\Gamma\left(\frac{1}{2} - \frac{\gamma}{n}\right) \Gamma\left(1 - \frac{\gamma}{n}\right)} - \\ &- \frac{\sqrt{\pi} Z^{-\gamma}}{4n\pi} \sum_{i=-i} \frac{1}{i} {}_5E_2 \left[\begin{matrix} \frac{1}{2} + \frac{1}{2}m + \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}m - \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}n - \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m, \frac{\gamma}{n} : Z^n e^{i\pi} \\ \frac{1}{2}, \frac{\gamma}{n} + 1 \end{matrix} \right]. \end{aligned} \quad (9)$$

$$\begin{aligned} \int_0^Z y^{n/2-\gamma-1} e^{1/2y^n} K_u\left(\frac{1}{2}y^n\right) dy &= \frac{\cos u\pi}{n\sqrt{\pi}} \Gamma\left(\frac{1}{2} + u - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2} - u - \frac{\gamma}{n}\right) \Gamma\left(\frac{\gamma}{n}\right) - \\ &- \frac{Z^{-\gamma} \cos u\pi}{n\sqrt{\pi}} {}_3E_1 \left[\begin{matrix} \frac{1}{2} + u, \frac{1}{2} - u, \frac{\gamma}{n} : Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \end{aligned} \quad (10)$$

$$\begin{aligned} \int_0^Z y^{-\gamma-1} K_n(y^{l/2}) K_m(e^{i\pi} y^{l/2}) dy &= \frac{\cos \pi \left(\frac{1}{2}m + \frac{1}{2}n\right) \cos \pi \left(\frac{1}{2}m - \frac{1}{2}n\right)}{2i\sqrt{\pi}} - \\ &- \left[\frac{1}{n} \cdot \Gamma\left(\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m - \frac{2\gamma+l}{2l}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}n + \frac{1}{2}m - \right. \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{2\gamma+l}{2l} \Big) \Gamma\left(\frac{1}{2} + \frac{1}{2}n - \frac{1}{2}m - \frac{2\gamma+l}{2l}\right) \Gamma\left(\frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m - \right. \\
& \left. - \frac{2\gamma+l}{2l}\right) \Gamma\left(\frac{2\gamma+l}{2l}\right) / \Gamma\left(\frac{1}{2} - \frac{2\gamma+l}{2l}\right) - \frac{1}{n} Z^{-\frac{2\gamma+l}{2}} \\
& E \left[\frac{1}{2} + \frac{1}{2}n + \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}n + \frac{1}{2}m, \frac{1}{2} + \frac{1}{2}n - \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m, \right. \\
& \left. \frac{1}{2}, \right. \\
& \left. \frac{2\gamma+l}{2l} : e^{i\pi} Z^n \right] \Bigg] + \frac{i \cos \frac{1}{2} \pi (1+m+n) \sin \frac{1}{2} \pi (n-m)}{2 \sqrt{\pi}} \\
& \frac{\frac{2\gamma+l}{2l} + 1}{\left. \right]} \\
& \left[\frac{1}{n} \cdot \Gamma\left(1 + \frac{1}{2}n + \frac{1}{2}m - \frac{\gamma+l}{l}\right) \Gamma\left(1 - \frac{1}{2}n + \frac{1}{2}m - \frac{\gamma+l}{l}\right) \right. \\
& \Gamma\left(1 + \frac{1}{2}n - \frac{1}{2}m - \frac{\gamma+l}{l}\right) \Gamma\left(1 - \frac{1}{2}n - \frac{1}{2}m - \right. \\
& \left. - \frac{\gamma+l}{l}\right) \Gamma\left(\frac{\gamma+l}{2}\right) / \Gamma\left(\frac{3}{2} - \frac{\gamma+l}{l}\right) - \frac{1}{n} Z^{-(\gamma+l)} \\
& E \left[1 + \frac{1}{2}n + \frac{1}{2}m, 1 + \frac{1}{2}n - \frac{1}{2}m, 1 - \frac{1}{2}n + \frac{1}{2}m, 1 - \frac{1}{2}n - \frac{1}{2}m, \frac{\gamma+l}{l} : e^{i\pi} Z^n \right] \Bigg] \\
& \left[\frac{3}{2}, \frac{\gamma+l}{l} + 1 \right] \Bigg] .
\end{aligned}$$

(11)

They will be deduced from two subsidiary theorem which will be stated and proved in section 2. It may be noted that the formulae hold provided the integrals are convergent.

The E-function is MacRobert's E-function where definition and properties are to be found in ([1], pp. 348–358 and [2], pp. 203–206). If $p \leq q$, the E-function is defined to be :

$$E \left[\begin{matrix} p; \alpha_r : Z \\ q; \beta_s \end{matrix} \right] = \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_p)}{\Gamma(\beta_1) \dots \Gamma(\beta_q)} {}_pF_q \left[\begin{matrix} \alpha_p : -1/Z \\ \beta_q \end{matrix} \right], \quad (12)$$

while if $p \geq q + 1$, $|\arg Z| < \pi$, then the E-function is

$$E \left[\begin{matrix} p; \alpha_r : Z \\ q; \beta_s \end{matrix} \right] = \sum_{r=1}^p \frac{\prod_{s=1}^{p'} \Gamma(\alpha_s - \alpha_r)}{\prod_{t=1}^q \Gamma(\beta_t - \alpha_r)} \Gamma(\alpha_r) Z^{\alpha_r} E_{p-1} \left[\begin{matrix} \alpha_r, \alpha_r - \beta_1 + 1, \dots, \\ \alpha_r - \alpha_1 + 1, \dots, \\ \alpha_r - \beta_q + 1 : (-1)^{p-q} Z \\ , \dots, \alpha_r - \alpha_p + 1 \end{matrix} \right]. \quad (13)$$

The following properties of the E-function then hold [4] :

$$Z^{2u} K_{2v}(Z e^{1/4\pi i}) K_2(Z e^{-1/4\pi i}) = 2^{3u-4} \pi^{-\frac{3}{2}} \sum_{i, -i} \frac{1}{i}$$

$${}_5E_0 \left[\frac{1}{2} u + v, \frac{1}{2} u - v, \frac{1}{2} u, \frac{1}{2} u + \frac{1}{2}, 1 : e^{i\pi} \frac{Z^4}{64} \right]. \quad (14)$$

$$Z^u K_v^2(Z) = 2^{-2} \pi^{\frac{1}{2}} \sum_{i, -i} \frac{1}{i} {}_4E_1 \left[\begin{matrix} v + \frac{1}{2} u, -v + \frac{1}{2} u, \frac{1}{2} u, 1 : e^{i\pi} Z^2 \\ \frac{1}{2} u + \frac{1}{2} \end{matrix} \right]. \quad (15)$$

$$Z^u K_v(Z) = 2^{u-2} \pi^{-1} \sum_{i, -i} \frac{1}{i} {}_3E_0 \left[\frac{1}{2} u + \frac{1}{2} v, \frac{1}{2} u - \frac{1}{2} v, 1 : e^{i\pi} \cdot \frac{1}{4} Z^2 \right]. \quad (16)$$

$$K_{2(b-a)}(2^{3/2} Z^{1/4} e^{1/4\pi i}) K_{2(b-a)}(2^{3/2} Z^{1/4} e^{-1/4\pi i}) = 2^{-4} \pi^{-3/2} Z^{-a}$$

$$\sum_{i, -i} \frac{1}{i} {}_5E_0 \left(1, a, b, a + \frac{1}{2}, 2a - b : Z e^{i\pi} \right). \quad (17)$$

$$Z^{l+\frac{1}{2}} \pi^{-\frac{1}{2}} e^{-\frac{1}{2}Z} K_m\left(\frac{1}{2}Z\right) = (2\pi)^{-\frac{1}{2}(n-1)} n^{l+\frac{1}{2}} \cdot \frac{1}{2\pi} \sum_{i=-i} \frac{1}{i} \\ E \left[\begin{matrix} \Delta\left(n; \frac{1}{2} + m + l\right), \Delta\left(n; \frac{1}{2} - m + l\right) : (Z/n)^n e^{i\pi} \\ \Delta(n; 1 + l) \end{matrix} \right], \quad (18)$$

where n is any positive integer, the symbol $\Delta(n; \alpha)$ say, has the same meaning as in (5).

$$Z^{l+\frac{1}{2}} \pi^{-\frac{1}{2}} e^{-\frac{1}{2}Z} K_m\left(\frac{Z}{2}\right) = \frac{1}{2\pi} \sum_{i=-i} \frac{1}{i} {}_3E_1 \left[\begin{matrix} \frac{1}{2} + m + l, \frac{1}{2} - m + l, 1 : Z e^{i\pi} \\ 1 + l \end{matrix} \right]. \quad (19)$$

$$\left(\frac{Z}{2}\right)^l \frac{Z}{\pi} K_m\left(\frac{Z}{2}\right) K_m\left(\frac{Z}{2}\right) = \pi^{-\frac{1}{2}} (2\pi)^{-n} n^{\frac{1}{2}+2l} \sum_{i=-i} \frac{1}{i} \\ E \left[\begin{matrix} \Delta\left(n; \frac{1}{2} + \frac{1}{2}l\right), \Delta\left(n; \frac{1}{2}l + m + \frac{1}{2}\right), \Delta\left(n; \frac{1}{2}l - m + \frac{1}{2}\right) : \left(\frac{Z}{4\pi^2}\right)^n e^{i\pi} \\ \Delta\left(n; 1 - \frac{1}{2}l\right) \end{matrix} \right]. \quad (20)$$

$$K_n(Z) = \frac{1}{4\pi} \sum_{i=-i} \frac{1}{i} {}_3E_0 \left[1, \frac{1}{2}n, -\frac{1}{2}n : \frac{1}{2}Z^2 e^{i\pi} \right]. \quad (21)$$

$$K_m(Z) K_n(Z) = \frac{\sqrt{\pi}}{4Z\pi} \sum_{i=-i} \frac{1}{i}$$

$${}_4E_1 \left[\begin{matrix} \frac{1}{2} + \frac{1}{2}m + \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}m - \frac{1}{2}n, \frac{1}{2} + \frac{1}{2}n - \frac{1}{2}m, \frac{1}{2} - \frac{1}{2}n - \frac{1}{2}m : e^{i\pi} Z^2 \\ \frac{1}{2} \end{matrix} \right]. \quad (22)$$

$$\cos n\pi E\left[\frac{1}{2} + n, \frac{1}{2} - n : 2Z\right] = (2\pi Z)^{\frac{1}{2}} e^Z K_n(Z). \quad (23)$$

$$K_n(Z) K_m(e^{i\pi} Z) = \frac{\cos \pi \left(\frac{1}{2} m + \frac{1}{2} n\right) \cos \pi \left(\frac{1}{2} m - \frac{1}{2} n\right)}{2iZ\sqrt{\pi}}$$

$$E\left(\frac{\frac{1}{2} + \frac{1}{2} m + \frac{1}{2} n, \frac{1}{2} + \frac{1}{2} m - \frac{1}{2} n, \frac{1}{2} - \frac{1}{2} m + \frac{1}{2} n; \frac{1}{2} - \frac{1}{2} m - \frac{1}{2} n : e^{i\pi} Z^2}{\frac{1}{2}}\right) +$$

$$+ i \frac{\cos \frac{1}{2} \pi (1 + m + n) \sin \frac{1}{2} \pi (n - m)}{2Z^2\sqrt{\pi}}$$

$$E\left(\frac{1 + \frac{1}{2} m + \frac{1}{2} n, 1 + \frac{1}{2} m - \frac{1}{2} n, 1 - \frac{1}{2} m + \frac{1}{2} n, 1 - \frac{1}{2} m - \frac{1}{2} n : e^{i\pi} Z^2}{\frac{3}{2}}\right).$$

(24)

$$K_{\pm \frac{1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z}. \quad (25)$$

§ 2. The theorem to be proved is

$$\int_0^x y^{-\gamma-1} E\left[\begin{matrix} p; \alpha_r : y^n \\ q; \beta_s \end{matrix}\right] dy = \frac{1}{n} \cdot \frac{\prod_{r=1}^p \Gamma\left(\alpha_r - \frac{\gamma}{n}\right)}{\prod_{s=1}^q \Gamma\left(\beta_s - \frac{\gamma}{n}\right)} \Gamma\left(\frac{\gamma}{n}\right) - \frac{1}{n} x^{-\gamma}$$

$${}_{p+1}E_{q+1}\left[\begin{matrix} p; \alpha_r; \frac{\gamma}{n} : x^n \\ q; \beta_s, \frac{\gamma}{n} + 1 \end{matrix}\right], \quad (26)$$

where $R\left(\alpha_r - \frac{\gamma}{n}\right) > 0$, n is any positive integer.

To prove (26) by (13) L.H.S. of (26) =
$$\sum_{r=0}^p \frac{\prod_{s=1}^{p'} \Gamma(\alpha_s - \alpha_r)}{\prod_{t=1}^q \Gamma(\beta_t - \alpha_r)} \Gamma(\alpha_r)$$

$$\sum_{n=0}^{\infty} \frac{(\alpha_r)_m (\alpha_r - \beta_1 + 1)_m \dots (\alpha_r - \beta_q + 1)_m}{m! (\alpha_r - \alpha_1 + 1)_m \dots (\alpha_r - \alpha_p + 1)_m} (-1)^{(p-q)m}$$

$$\int_0^x y^{-\gamma + n\alpha_r + nm - 1} dy = -\frac{X^{-\gamma}}{n} \sum_{r=1}^p \frac{\prod_{s=1}^{p+1} \Gamma(\alpha_s - \alpha_r)}{\prod_{t=1}^{q+1} \Gamma(\beta_t - \alpha_r)} \Gamma(\alpha_r) X^{n\alpha_r}$$

$${}_{q+2}F_p \left[\begin{matrix} \alpha_r, \alpha_r - \beta_1 + 1, \dots, \alpha_r - \beta_q + 1, \alpha_r - \frac{\gamma}{n} : (-1)^{p-q} x^n \\ \alpha_r - \alpha_1 + 1, \dots, \alpha_r - \alpha_p + 1, \alpha_r - \frac{\gamma}{n} + 1 \end{matrix} \right] = \text{R.H.S.}$$

Now writing λx^n for x^n in (26) and also λy^n for y^n in the same formula where $\lambda = e^{i\pi}$, it follows that

$$\begin{aligned} \int_0^x y^{-\gamma-1} \sum_{i=-i} \frac{1}{i} E \left[\begin{matrix} p; \alpha_r : e^{i\pi} y^n \\ q; \beta_s \end{matrix} \right] dy &= \frac{2\pi}{n} \cdot \frac{\prod_{r=1}^p \Gamma \left(\alpha_r - \frac{\gamma}{n} \right)}{\prod_{t=1}^q \Gamma \left(\beta_t - \frac{\gamma}{n} \right) \Gamma \left(1 - \frac{\gamma}{n} \right)} - \\ &- \frac{x^{-\gamma}}{n} \sum_{i=-i} \frac{1}{i} {}_{p+1}E_{q+1} \left[\begin{matrix} p; \alpha_r, \frac{\gamma}{n} : e^{i\pi} x^n \\ q; \beta_s, \frac{\gamma}{n} + 1 \end{matrix} \right]. \end{aligned} \quad (27)$$

Then (27) in combination with (14) gives (1);
 (27) in combination with (15) gives (2);
 (27) in combination with (16) gives (3);
 (27) in combination with (17) gives (4);
 (27) in combination with (18) gives (5);
 (27) in combination with (19) gives (6);
 (27) in combination with (20) gives (7);
 (27) in combination with (21) gives (8);

(27) in combination with (22) gives (9);

(26) in combination with (23) gives (10);

(26) in combination with (24) gives (11).

Some special cases :

From (1) and (25) we deduce

$$\int_0^Z y^{\frac{nu}{2} - \frac{n}{4} - \gamma - 1} e^{2^{-3} y^{-\frac{n}{2}}} dy = \frac{1}{n\sqrt{2\pi}\sqrt{\pi}} \Gamma\left(\frac{1}{2}u + \frac{1}{4} - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2}u - \frac{1}{4} - \frac{\gamma}{n}\right) \\ \Gamma\left(\frac{1}{2}u - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2}u + \frac{1}{2} - \frac{\gamma}{n}\right) - \frac{Z^{-\gamma}}{n^{3/2} \pi^{5/2}} \sum_{i, -i} \frac{1}{i} \\ {}_4E_1 \left[\begin{matrix} \frac{1}{2}u + \frac{1}{4}, \frac{1}{2}u - \frac{1}{4}, \frac{1}{2}u, \frac{1}{2}u + \frac{1}{2}, 1, \frac{\gamma}{n} : e^{i\pi} Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (28)$$

From (3) and (25) we deduce

$$\int_0^Z e^{-2y^{\frac{n}{2} + \frac{nu}{2} - \gamma - 1} y^{\frac{n}{4}}} dy = \frac{1}{\pi^{\frac{1}{2}} n} \Gamma\left(\frac{1}{2}u + \frac{1}{4} - \frac{\gamma}{n}\right) \Gamma\left(\frac{1}{2}u - \frac{1}{4} - \frac{\gamma}{n}\right) - \\ - \frac{Z^{-\gamma}}{2n\pi\sqrt{\pi}} \sum_{i, -i} \frac{1}{i} {}_4E_1 \left[\begin{matrix} \frac{1}{2}u + \frac{1}{4}, \frac{1}{2}u - \frac{1}{4}, 1, \frac{\gamma}{n} : e^{i\pi} Z^n \\ \frac{\gamma}{n} + 1 \end{matrix} \right]. \quad (29)$$

From (3) and (16) we deduce

$$\int_0^Z t^\nu K_{\nu-1}(t) dt = 2^{\nu-1} \Gamma(\nu) - Z^\nu K_\nu(Z), \quad (30)$$

$$R(\nu) > 0.$$

(30) is ([5], p. 126 (51)).

In (5), we put $Z \rightarrow \infty$ we deduce

$$\int_0^\infty e^{-t} t^u K_\nu(t) dt = \frac{\Gamma(u + \nu + 1) \Gamma(u - \nu + 1)}{2^u \left(\frac{3}{2}\right)_u}, \quad (31)$$

$$R(u \pm \nu) > -1.$$

(31) is ([5], p. 107 (4)).

Writing $n = 1$ in (1) to (10) we get formulae from (32) to (41) in [3]. We can deduce infinite integrals for modified Bessel functions of the second kind of the previous formulae by putting $Z \rightarrow \infty$.

The asymptotic expansions for generalized hypergeometric function and E -function for large and small values of argument were evaluated in [4].

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ON SOME ASYMPTOTIC PROPERTIES OF THE A POSTERIORI DISTRIBUTION

LUMINIȚA STATE

Let $(\Omega, \mathcal{F})$ and $(X, \mathcal{A})$ be two measurable spaces. Assume that μ is a σ -finite measure on $\mathcal{A}$. Let $\{z_n; n \geq 1\}$ be a sequence of random elements defined on $(\Omega, \mathcal{F})$, independent and identically distributed. Suppose that Θ is an open subset of R^k endowed with the usual topology, and for any θ from Θ , P_θ is a probability measure on $\mathcal{A}$.

Throughout this paper it is assumed that the following conditions are fulfilled:

A1. The random element z_1 is distributed according to one of the probability laws P_θ on $(X, \mathcal{A})$, where $\theta \in \Theta$. For simplicity sake, we also denote by P_θ the joint distribution of the sequence $\{z_n; n \geq 1\}$ on $(X^\infty, \mathcal{A}^\infty)$, if the distribution law of z_1 is P_θ .

A2. $P_{\theta_1} = P_{\theta_2} \Rightarrow \theta_1 = \theta_2$.

As. For every θ from Θ , P_θ is dominated by the σ -finite measure μ . We denote by $f(z, \theta)$ the density $dP_\theta/d\mu$ of z_1 , when θ is the „true” value of the parameter. Suppose that for all $z \in X$, $\Phi(z, \theta)$ defined by $\Phi(z, \theta) = \log f(z, \theta)$ if finite and continuous on Θ . As usual, we will consider $\log 0 = -\infty$ and $\theta \log 0 = 0$.

A4. For almost all z , $\partial \Phi(z, \theta)/\partial \theta_i$, and $\partial^2 \Phi(z, \theta)/\partial \theta_i \partial \theta_j$; $i, j = 1, \dots, k$ exist and are continuous on Θ .

A5. For every $\theta \in \Theta$, there exists $\varepsilon(\theta) > 0$, such that for any $i, j = 1, \dots, k$,

$$E_\theta \left\{ \sup_{\{||s-\theta|| < \varepsilon(\theta)\}} \frac{\partial^2 \Phi(z, s)}{\partial s_i \partial s_j} \right\} < \infty$$

where E_θ as usual, denotes that the computation is carried out when θ is the „true” value of the parameter.

A6. For every $\theta \in \Theta$, let $A_{ij}(\theta)$ be given by:

$$A_{ij}(\theta) = E_\theta \left\{ \frac{\partial^2 \Phi(z, \theta)}{\partial \theta_i \partial \theta_j} \right\}$$

and $A(\theta)$ be the $k \times k$ matrix whose entries are $A_{ij}(\theta)$. We assume that for any θ from Θ , the matrix $-A(\theta)$ is positive definite.

A7. For any $\theta \in \Theta$ and for any $\varepsilon > 0$,

$$E_\theta \left\{ \sup_{\{||s-\theta|| > \varepsilon\}} (\Phi(z, s) - \Phi(z, \theta)) \right\} < 0$$

A8. Let Λ be a measure on Θ endowed with the Borel σ -field. Assume that Λ is dominated by ν , the Lebesgue measure on R^k , and there exists a measurable version λ of the Radon-Nicodym derivative $d\Lambda/d\nu$, such that the following conditions are fulfilled:

(i) λ is continuous on Θ .

(ii) There exists $M < \infty$ such that $0 < \lambda(\theta) < M$ for all θ from Θ .

(iii) There exists $m \geq 2$, such that $\int \|\theta\|^m \lambda(\theta) d\theta < \infty$. We will call the measure Λ , the a priori distribution on Θ , and λ the a priori density on Θ .

For each n , $z_1, z_2, \dots, z_n$ define $\hat{\theta}_n(z_1, z_2, \dots, z_n)$ to be a value of θ , such that:

$$(1) \quad \prod_{i=1}^n f(z_i, \hat{\theta}_n(z_1, z_2, \dots, z_n)) = \sup_{\theta \in \Theta} \prod_{i=1}^n f(z_i, \theta).$$

It can be shown, that if the conditions A1–A8 are fulfilled, then, there exists a measurable version of $\hat{\theta}_n = \hat{\theta}_n(z_1, z_2, \dots, z_n)$ such that $\hat{\theta}_n \rightarrow \theta_n$ a.s. P_θ , for all $\theta \in \Theta$, when the „true” value of the parameter is θ .

The random vector $\hat{\theta}_n$ is called the maximum likelihood estimator of the parameter θ .

Fix $\theta_0 \in \Theta$; we will assume throughout this paper that the „true” value of the parameter is θ_0 .

For any $\theta \in \Theta$, n , $z_1, z_2, \dots, z_n$ we define:

$$(2) \quad \lambda(\theta; n) = \begin{cases} \frac{\lambda(\theta) \prod_{i=1}^n f(z_i, \theta)}{\int \prod_{i=1}^n f(z_i, s) \lambda(s) ds}, & \text{if } \int \prod_{i=1}^n f(z_i, s) \lambda(s) ds \neq 0 \\ 0, & \text{if } \int \prod_{i=1}^n f(z_i, s) \lambda(s) ds = 0 \end{cases}$$

For every n , $z_1, z_2, \dots, z_n$ we consider P_n defined on the Borel σ -field $\mathcal{B}$ of Θ , given by:

$$(3) \quad P_n(A) = \int_A \lambda(\theta; n) d\theta$$

Obviously, for each $n \geq 1$, P_n is a probability measure on $\mathcal{B}$ a.s.— P_{θ_0} .

We will call P_n the a posteriori distribution on θ ; $\lambda(\theta; n)$ the a posteriori density on Θ .

Lemma 1 [1] Suppose assumptions A1 – A8 hold. Then, for any δ, α such that $\delta > 0, \alpha > 0$, and for any consistent estimator $\tilde{\theta}_n$ of the parameter θ ,

$$(4) \quad \int_{\{\|\theta - \tilde{\theta}_n\| \geq \delta\}} \|\theta - \tilde{\theta}_n\|^m \lambda(\theta; n) d\theta = o(n^{-\alpha}).$$

For every $t \in R^k, n \geq 1$ and $z_1, \dots, z_n$, we define:

$$(5) \quad v_n(t) = \frac{\prod_{i=1}^n f\left(z_i, \hat{\theta}_n + \frac{t}{\sqrt{n}}\right)}{\prod_{i=1}^n f(z_i, \hat{\theta}_n)}$$

and

$$(6) \quad H(t, n) = |v_n(t) - (2\pi)^{-\frac{k}{2}} \det(-A^{-1}(\theta_0))^{\frac{1}{2}} \Phi(-A^{-1}(\theta_0), t)|$$

where $\Phi(B, t)$ represents the density of the multivariate normal distribution with mean 0 (an $1 \times k$ vector) and the covariance matrix B .

Using lemma 2.1.[1] one can show the following result:

Lemma 2. Suppose assumptions A1–A8 hold. Then, there exists $\delta^* > 0$, such that:

$$(7) \quad \int_{\{\|t\| < \delta^* \sqrt{n}\}} (1 + \|t\|^m) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) H(t, n) dt \rightarrow 0, \text{ as } n \rightarrow \infty,$$

a.s. — P_{θ_0} .

Moreover, if $\|t\| < \delta^* \sqrt{n}$, then $P_{\theta_0}(\lim_{n \rightarrow \infty} H(t, n) = 0) = 1$.

Using (2.40) [2], we can prove the following lemma:

Lemma 3 There exists $\delta^* > 0$ and a positive definite matrix W , such that for any $t \in R^k$, if $\|t\| \leq \delta^* \sqrt{n}$, then, $v_n(t) \leq \exp(-tWt')$, where' denotes transpose.

Moreover, the matrix W can be taken as $-\frac{1}{2}(A(\theta_0) + \varepsilon_0 I)$, where $\varepsilon > 0$, and I represents the unity matrix.

Let z be $(z_1, z_2, \dots)$

Lemma 4[2]. Under assumptions A1–A8, for all $\delta > 0$, there exist $\varepsilon(\delta) > 0$, and $M(z) < \infty$, such that for all $n > M(z)$,

$$(8) \quad \sup_{\{\|t\| \geq \delta/\sqrt{n}\}} v_n(t) \leq \exp(-n \varepsilon(\delta)).$$

$$\begin{aligned}
 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} ; n \right) &= \frac{\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \prod_{i=1}^n f \left(z_i, \hat{\theta}_n + \frac{t}{\sqrt{n}} \right)}{\int \prod_{i=1}^n f(z_i, s) \lambda(s) ds} = \\
 &= \frac{\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \frac{\prod_{i=1}^n f \left(z_i, \hat{\theta}_n + \frac{t}{\sqrt{n}} \right)}{\prod_{i=1}^n f(z_i, \hat{\theta}_n)}}{\int \prod_{i=1}^n \frac{f(z_i, s)}{f(z_i, \hat{\theta}_n)} \lambda(s) ds} = \\
 &= \frac{\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) v_n(t)}{\frac{1}{n^{k/2}} \int v_n(s) \lambda \left(\hat{\theta}_n + \frac{s}{\sqrt{n}} \right) ds}
 \end{aligned}$$

Therefore, we get :

$$(9) \quad \frac{\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} ; n \right)}{n^{k/2}} = \frac{\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) v_n(t)}{\int v_n(s) \lambda \left(\hat{\theta}_n + \frac{s}{\sqrt{n}} \right) ds}$$

Let c_n be the following random variable :

$$(10) \quad c_n = \int v_n(s) \lambda \left(\hat{\theta}_n + \frac{s}{\sqrt{n}} \right) ds$$

Then, one can prove :

$$\text{Lemma 5 } P_{\theta_0} \left(\lim_{n \rightarrow \infty} c_n = \lambda(\theta_0) (2\pi)^{\frac{k}{2}} \det(-A^{-1}(\theta_0))^{\frac{1}{2}} \right) = 1 \quad (11)$$

Let δ_{θ_0} be the distribution degenerate at θ_0 .

In the following it is considered the problem of weak convergence of the sequence P_n to δ_{θ_0} . More specifically, it can be shown the following theorem :

Theorem 1 The sequence P_n converges weakly to the distribution degenerate at θ_0 , a.s. P_{θ_0} , i.e. $P_{\theta_0}(P_n \xrightarrow{} \delta_{\theta_0}) = 1$.*

Proof Using the theorem of Billingsley-Alexandroff [8] p. 11, if we prove that a.s. $-P_{\theta_0}$ for every closed set F , the relation $\limsup_{n \leftarrow \infty} P_n(F) \leq \delta_{\theta_0}(F)$

holds, then $P_n \xrightarrow{*} \delta_{\theta_0}$ a.s. $-P_{\theta_0}$. Let F be a closed subset of R^k endowed with the usual topology. If $\theta_0 \in F$, then $\delta_{\theta_0}(F) = 1$, hence since P_n are pro-

bability measures, we get $\limsup_{n \rightarrow \infty} P_n(F) \leq \delta_{\theta_0}(F) = 1$ for every $z = (z_1, z_2, \dots)$ from X^∞ .

If $\theta_0 \notin F$, then $\delta_{\theta_0}(F) = 0$. There exists $\delta_0 > 0$ such that

$$\{\theta / \|\theta - \theta_0\| < 2\delta_0\} \cap F = \emptyset$$

Since $P_{\theta_0}(\lim_{n \rightarrow \infty} \hat{\theta}_n = \theta_0) = 1$, let $N \in \mathcal{A}^\infty$ be such that $P_{\theta_0}(N) = 0$, and $z \notin N$ implies $\hat{\theta}_n \rightarrow \theta_0$.

Let $z \notin N$. There exists $M(z, \delta_0)$, such that for any $n \geq M(z, \delta_0)$, $\|\hat{\theta}_n - \theta_0\| < \delta_0$.

It is obvious that if θ is such that $\|\theta - \hat{\theta}_n\| < \delta_0$, then $\|\theta - \theta_0\| < 2\delta_0$. Therefore, $\{\theta / \|\theta - \hat{\theta}_n\| < \delta_0\} \subset \{\theta / \|\theta - \theta_0\| < 2\delta_0\}$, hence, $\{\theta / \|\theta - \hat{\theta}_n\| < \delta_0\} \cap F = \emptyset$. We get $F \subset \{\theta / \|\theta - \hat{\theta}_n\| \geq \delta_0\}$.

Using lemma 1 with $\hat{\theta}_n$ as $\tilde{\theta}_n$ and $\delta = \delta_0$, we obtain that for any $\alpha > 0$,

$$\int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \|\theta - \hat{\theta}_n\|^m \lambda(\theta; n) d\theta = o(n^{-\alpha}).$$

$$\text{But, } \int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \|\theta - \hat{\theta}_n\|^m \lambda(\theta; n) d\theta \geq \delta_0^m \int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \lambda(\theta; n) d\theta$$

Therefore, we get: $\int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \lambda(\theta; n) d\theta = o(n^{-\alpha})$ for any $\alpha > 0$, hence

$$\limsup_{n \rightarrow \infty} \int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \lambda(\theta; n) d\theta = 0$$

But, since $0 \leq \int_F \lambda(\theta; n) d\theta \leq \int_{\{\|\theta - \hat{\theta}_n\| \geq \delta_0\}} \lambda(\theta; n) d\theta$, we get:

$$\limsup_{n \rightarrow \infty} \int_F \lambda(\theta; n) d\theta = 0.$$

Therefore, $\limsup_{n \rightarrow \infty} P_n(F) \leq \delta_{\theta_0}(F)$ for any F closed subset of R^k and for any $z \in N$, hence $P_{\theta_0}(P_n^* \delta_{\theta_0}) = 1$.

In the following, we will consider the asymptotic behaviour of the informational energy [6] corresponding to the a posteriori distribution. Let $E_n(z)$ denotes the informational energy of the a posteriori distribution

As it is well known, $E_n(z)$ will be given by : $E_n(z) = \int \lambda^2(\theta; n) d\theta$.

Theorem 2. Suppose that conditions A1–A8 hold. Then,

$$P_{\theta_0} \left(\lim_{n \rightarrow \infty} \frac{E_n(z)}{n^{k/2}} = \frac{1}{2^k \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2}} \right) = 1.$$

Proof

Let $\delta > 0$ be such that lemma 3 holds. Then,

$E_n(z) = E_n^{(1)}(z, \delta) + E_n^{(2)}(z, \delta)$, where $E_n^{(1)}(z, \delta)$, and $E_n^{(2)}(z, \delta)$ are given by :

$$(12) \quad E_n^{(1)}(z, \delta) = \int_{\{\|\theta - \hat{\theta}_n\| \leq \delta\}} \lambda^2(\theta; n) d\theta$$

$$(13) \quad E_n^{(2)}(z, \delta) = \int_{\{\|\theta - \hat{\theta}_n\| > \delta\}} \lambda^2(\theta; n) d\theta$$

$$\begin{aligned} E_n^{(1)}(z, \delta) &= \int_{\{\|\theta - \hat{\theta}_n\| \leq \delta\}} \lambda^2(\theta; n) d\theta = \frac{1}{n^{k/2}} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}; n\right) dt = \\ &= \frac{n^{k/2}}{c_n^2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt \end{aligned}$$

where c_n is given by (10), and we used (9) and the transformation $\theta \rightarrow \hat{\theta}_n + \frac{t}{\sqrt{n}}$.

Let $I_n(z, \delta)$ be given by :

$$(14) \quad \begin{aligned} I_n(z, \delta) &= \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) H(t, n) dt = \\ &= \int v_n(t) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) H(t, n) dt. \end{aligned}$$

Using lemma 3, we get :

$$v_n(t) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \leq \exp(-t W t')$$

$$H(t, n) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} = |v_n(t) \chi_{\{\|t\| \leq \sqrt{n}\}} - (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2}.$$

$$\chi_{\{\|t\| \leq \delta \sqrt{n}\}} \Phi(-A^{-1}(\theta_0), t) \leq \\ \chi_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) + \exp\left(\frac{1}{2} t A(\theta_0) t'\right)$$

Since the matrix W can be taken as $-\frac{1}{2}(A(\theta_0) + \varepsilon_0 I)$ with $\varepsilon_0 > 0$, we obtain :

$$-t W t' + \frac{1}{2} t A(\theta_0) t' = \frac{1}{2} t (A(\theta_0) + \varepsilon_0 I) t' + \frac{1}{2} t A(\theta_0) t' = t (A(\theta_0) + \frac{\varepsilon_0}{2} I) t'$$

Since the matrix $A(\theta_0) + \varepsilon_0 I$ is negative definite, $A(\theta_0) + \frac{\varepsilon_0}{2} I$ is negative definite, therefore $\exp(-t W t' + \frac{1}{2} t A(\theta_0) t')$ is integrable, over R^k

It is obvious that $v_n(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} H(t, n)$ is dominated by the integrable function

$$M^2 (\exp(-2t W t') + \exp(-t W t' + \frac{1}{2} t A(\theta_0) t'))$$

Using lemma 2, we obtain :

$$P_{\theta_0}(\lim_{n \rightarrow \infty} v_n(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) H(t, n) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} = 0) = 1.$$

Using the bounded convergence theorem, we get :

$$(15) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} I_n(z, \delta) = 0) = 1.$$

From (15) we infer :

$$(16) \quad \lim_{n \rightarrow \infty} \left\{ \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt - \right. \\ \left. \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \Phi(-A^{-1}(\theta_0), t) dt \right\} = \\ = 0, \text{ a.s. } - P_{\theta_0}.$$

Let $I_n^{(1)}(z, \delta)$ be given by

$$(17) \quad I_n^{(1)}(z, \delta) = (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right)$$

$$\Phi(-A^{-1}(\theta_0), t) dt =$$

$$\int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) dt$$

We also consider $I_n^{(2)}(z, \delta)$ defined by :

$$(18) \quad I_n^{(2)}(z, \delta) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) H(t, n) dt.$$

Since,

$$\begin{aligned} & \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \exp \left(\frac{1}{2} t A(\theta_0) t' \right) H(t, n) \leq \\ & M^2 \left(\exp \left(\frac{1}{2} t A(\theta_0) t' \right) \left(\exp \left(\frac{1}{2} t A(\theta_0) t' + \frac{\varepsilon_0}{2} \|t\|^2 \right) + \exp \left(\frac{1}{2} t A(\theta_0) t' \right) \right) \right) = \\ & = M^2 \left(\exp \left(t A(\theta_0) t' + \frac{1}{2} \varepsilon_0 \|t\|^2 \right) + \exp \left(t A(\theta_0) t' \right) \right) = \\ & = M^2 \left(\exp (t(A(\theta_0) + \varepsilon_0 I) t') \exp \left(-\frac{1}{2} \varepsilon_0 \|t\|^2 \right) + \exp (t A(\theta_0) t') \right), \end{aligned}$$

and since $A(\theta_0)$ and $A(\theta_0) + \varepsilon_0 I$ are negative definite, we get that $\lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) H(t, n) \chi_{\{\|t\| \leq \delta \sqrt{n}\}}$ is dominated by an integrable function.

Using, $\lim_{n \rightarrow \infty} \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) H(t, n) = 0$ a.s. $- P_{\theta_0}$, and the bounded convergence theorem, we obtain :

$$(19) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} I_n^{(2)}(z, \delta) = 0) = 1.$$

From (19) we obtain :

$$(20) \quad \lim_{n \rightarrow \infty} \left\{ \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) dt - \right. \\ \left. - (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) \right. \\ \left. \Phi(-A^{-1}(\theta_0), t) dt \right\} = 0, \text{ a.s. } - P_{\theta_0}.$$

$$\text{But, } (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{2} t A(\theta_0) t' \right) \\ \Phi(-A^{-1}(\theta_0), t) dt = \\ \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp(t A(\theta_0) t') dt$$

Using $\lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp(t A(\theta_0) t') \leq M^2 \exp(t A(\theta_0) t')$, we infer that

$\lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp(t A(\theta_0) t') \chi_{\{\|t\| \leq \delta \sqrt{n}\}}$ is dominated by an integrable function. Since we assumed that λ is a continuous function on Θ , and $\hat{\theta}_n$ is consistent, we obtain :

$$\lim_{n \rightarrow \infty} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \exp(t A(\theta_0) t') = \\ \lambda^2(\theta_0) \exp(t A(\theta_0) t'), \text{ a.s. } - P_{\theta_0}.$$

Using again the bounded convergence theorem we get :

$$(21) \quad \lim_{n \rightarrow \infty} (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(-\frac{1}{2} t A(\theta_0) t' \right) \\ \Phi(-A^{-1}(\theta_0), t) dt =$$

$$\lambda^2(\theta_0) \int \exp\left(-\frac{1}{2}((\sqrt{2t})(-A(\theta_0))(\sqrt{2t})')\right) dt =$$

$$\frac{\lambda^2(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2}}{2^{k/2}} \text{ a.s.} - P_{\theta_0}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \exp\left(\frac{1}{2} t A(\theta_0) t'\right) dt =$$

$$= \lambda^2(\theta_0) \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \text{ a.s.} - P_{\theta_0}.$$

Finally, we get :

$$(22) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} I_n^{(1)}(z, \delta) = \lambda^2(\theta_0) \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2}) = 1$$

Using (16) and (22) we obtain :

$$(23) \quad P_{\theta_0}\left(\lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt =\right.$$

$$\left. = \lambda^2(\theta_0) \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2}\right) = 1.$$

Using lemma 5, we get :

$$(24) \quad P_{\theta_0}\left(\lim_{n \rightarrow \infty} \frac{E_n^{(1)}(z, \delta)}{n^{k/2}} = \frac{1}{2^k \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2}}\right) = 1.$$

We will consider in the following the quantity of informational energy contained by $R^k \setminus S(\hat{\theta}_n; \delta)$, where $S(\hat{\theta}_n; \delta)$ stands for the sphere with center $\hat{\theta}_n$ and radius δ , ie. $E_n^{(2)}(z, \delta)$,

$$E_n^{(2)}(z, \delta) = \int_{\{\|\theta - \hat{\theta}_n\| > \delta\}} \lambda^2(\theta; n) d\theta = n^{k/2} \int_{\{\|t\| > \delta \sqrt{n}\}} \frac{v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right)}{c_n^2} dt =$$

$$= \frac{n^{k/2}}{c_n^2} \int_{\{\|t\| > \delta \sqrt{n}\}} v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt.$$

Using lemma 4, we get :

$$\begin{aligned}
 E_n^{(2)}(z, \delta) &\leq \frac{n^{k/2}}{c_n^2} \exp(-2n\varepsilon(\delta)) \int_{\{\|t\| > \delta \sqrt{n}\}} \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt \leq \\
 &M \frac{n^{k/2}}{c_n^2} \exp(-2n\varepsilon(\delta)) \int_{\{\|t\| > \delta \sqrt{n}\}} \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt = \\
 &= M \frac{n^{k/2}}{c_n^2} \exp(-2n\varepsilon(\delta)) n^{k/2} \int_{\{\|\theta - \hat{\theta}_n\| > \delta\}} \lambda(\theta) d\theta \leq \\
 &\leq \frac{M n^k}{c_n^2} \exp(-2n\varepsilon(\delta)) \int \lambda(\theta) d\theta.
 \end{aligned}$$

Obviously, from $\int \|\theta\|^m \lambda(\theta) d\theta < \infty$, we infer

$$\int \lambda(\theta) d\theta = K < \infty.$$

Therefore, we obtain :

$$E_n^{(2)}(z, \delta) \leq MK \frac{n^k}{c_n^2} \exp(-2n\varepsilon(\delta)).$$

Using P_{θ_0} ($\lim_{n \rightarrow \infty} c_n = \lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} = 1$ and the assumption A8(ii), we finally get :

$$(25) \quad E_n^{(2)}(z, \delta) = 0(n^{-\alpha}) \text{ a.s. } - P_{\theta_0},$$

for any $\alpha > 0$, where the exceptional set of data sequences can depend on δ and α .

We infer that for any $\alpha > 0$,

$$E_n(z, \delta) - E_n^{(1)}(z, \delta) = 0(n^{-\alpha}) \text{ a.s. } - P_{\theta_0}.$$

From this, and (24) we finally get :

$$P_{\theta_0}\left(\frac{E_n(z, \delta)}{n^{k/2}}\right) \xrightarrow{n \rightarrow \infty} \frac{1}{2^k \pi^{k/2} \det(-A^{-1}(\theta_0))^{1/2}} = 1$$

Define the density of informational energy in $\hat{\theta}_n$ with respect to the a posteriori distribution by :

$$(26) \quad \rho_n(\hat{\theta}_n) = \limsup_{\delta \rightarrow 0} \frac{E_n^{(1)}(z, \delta)}{V_\delta^k}$$

where V_δ^k stands for the volume of the sphere with radius δ in R^k .

Theorem 3 Suppose that conditions A1–A8 hold. Then,

$$(27) \quad \varphi_n(\hat{\theta}_n) = \lambda^2(\hat{\theta}_n) \frac{n^k}{c_n^2}$$

$$(28) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} \frac{\varphi_n(\hat{\theta}_n)}{n^k} = \frac{1}{(2\pi)^k \det(-A^{-1}(\theta_0))} \right) = 1$$

Proof

Let $\delta^* > 0$ be given by lemma 3. Obviously, if $0 < \delta < \delta^*$, then, for this δ , lemma 3 also holds.

Let $0 < \delta < \delta^*$. Then :

$$V_\delta^k = \frac{2\delta^k \pi^{k/2}}{k \Gamma(k/2)}$$

We get the following series of equalities :

$$\begin{aligned} \frac{E_n^{(1)}(z, \delta)}{V_\delta^k} &= \frac{k \Gamma(k/2)}{2\delta^k \pi^{k/2}} \int_{\{\|\theta - \hat{\theta}_n\| \leq \delta\}} \lambda^2(\theta; n) \, d\theta = \\ &= \frac{k \Gamma(k/2) n^{k/2}}{2\delta^k \pi^{k/2}} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \frac{v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right)}{c_n^2} \, dt = \\ &= \frac{kn^{k/2} \Gamma(k/2)}{2\delta^k \pi^{k/2} c_n^2} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n^2(t) \lambda^2\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \, dt = \\ &= \frac{kn^{k/2} \Gamma(k/2) \delta^k}{2\delta^k \pi^{k/2} c_n^2} \int_{\{\|s\| \leq \sqrt{n}\}} v_n^2(s\delta) \lambda^2\left(\hat{\theta}_n + \delta \frac{s}{\sqrt{n}}\right) \, ds = \\ &= \frac{kn^{k/2} \Gamma(k/2)}{2\pi^{k/2} c_n^2} \int_{\{\|s\| \leq \sqrt{n}\}} v_n^2(s\delta) \lambda^2\left(\hat{\theta}_n + \delta \frac{s}{\sqrt{n}}\right) \, ds. \end{aligned}$$

For any n , since $\hat{\theta}_n$ is the maximum likelihood estimator,

$$v_n(s\delta) \leq 1, \text{ for any } s \in R^k \text{ and } \delta > 0.$$

Fix $n \geq 1$. Since $0 < \lambda(\theta) < M$ for any θ , we get that for any $\delta > 0$, $v_n^2(s\delta) \lambda^2\left(\hat{\theta}_n + \delta \frac{s}{\sqrt{n}}\right) \chi_{\{\|s\| \leq \sqrt{n}\}}$ is dominated by the integrable function

$$M^2 \chi_{\{\|s\| \leq \sqrt{n}\}}$$

Since n is fixed, $v_n(t)$ is a continuous function of t . Obviously, $\lim_{\delta \rightarrow 0} v_n(s\delta) =$

$= 1$. Also, λ is a continuous function and $\hat{\theta}_n$ is consistent, therefore

$$\lim_{\delta \rightarrow 0} \lambda^2\left(\hat{\theta}_n + \delta \frac{s}{\sqrt{n}}\right) = \lambda^2(\hat{\theta}_n).$$

Using the observation 1. p. 126[4], we get :

$$\lim_{\delta \rightarrow 0} \int \chi_{\{\|s\| \leq \sqrt{n}\}} v_n^2(s\delta) \lambda^2\left(\hat{\theta}_n + \delta \frac{s}{\sqrt{n}}\right) ds = \lambda^2(\hat{\theta}_n) V_{\sqrt{n}}^k.$$

Hence, there exists $\lim_{\delta \rightarrow 0} \frac{E_n^{(1)}(z, \delta)}{V_{\delta}^k}$ and

$$\lim_{\delta \rightarrow 0} \frac{E_n^{(1)}(z, \delta)}{V_{\delta}^k} = \frac{k n^{k/2} \Gamma(k/2)}{2 \pi^{k/2} c_n^2} \frac{2 n^{k/2} \pi^{k/2}}{k \Gamma(k/2)} \lambda^2(\hat{\theta}_n).$$

Therefore, for any $n \geq 1$, $\rho_n(\hat{\theta}_n) = \lambda^2(\hat{\theta}_n) \frac{n^k}{c_n^2}$.

From (27), using the consistency of the estimator $\hat{\theta}_n$, and (11) we finally get :

$$P_{\theta_0} \left(\lim_{n \rightarrow \infty} \rho_n(\hat{\theta}_n) \frac{1}{n^k} = \frac{1}{(2\pi)^k \det(-A^{-1}(\theta_0))} \right) = 1.$$

In the following, it will be considered the asymptotic behaviour of the entropy corresponding to the a posteriori distribution.

Let $H_n(\lambda; z)$ be the entropy of the a posteriori distribution, if λ represents the a priori density and the data sequence is represented by z .

$$(29) \quad H_n(\lambda; z) = - \int \lambda(\theta; n) \log_2 \lambda(\theta; n) d\theta.$$

Theorem 4 Suppose the assumptions A1–A8 hold. If $\lambda_1, \dots, \lambda_k$ are the eigenvalues of the matrix $(-A(\theta_0))$, then :

$$P_{\theta_0} \left(\lim_{n \rightarrow \infty} \left(H_n(\lambda; z) + \frac{k}{2} \log n \right) = \frac{k}{2^{k/2+2}} + \frac{k}{2} \log(2\pi) + \right. \\ \left. + \frac{1}{2} \sum_{j=1}^k \log \lambda_j \right) = 1.$$

Proof :

Let $\delta^* > 0$ be given by lemma 2, and $\delta^{**} > 0$ be given by lemma 3. Let $\tilde{\delta} = \min \{\delta^*, \delta^{**}\}$. It is obvious, that if $0 < \delta < \tilde{\delta}$, then for this δ , the lemma 2 and lemma 3 also hold.

Let $0 < \delta < \tilde{\delta}$. Then defining $H_n^{(1)}(\delta; \lambda; z)$ and $H_n^{(2)}(\delta; \lambda; z)$ by :

$$(30) \quad H_n^{(1)}(\delta; \lambda; z) = - \int_{\{||\theta - \hat{\theta}_n|| \leq \delta\}} \lambda(\theta; n) \log \lambda(\theta; n) d\theta$$

$$(31) \quad H_n^{(2)}(\delta; \lambda; z) = - \int_{\{||\theta - \hat{\theta}_n|| > \delta\}} \lambda(\theta; n) \log \lambda(\theta; n) d\theta, \text{ we get :}$$

$$(32) \quad H_n(\lambda; z) = H_n^{(1)}(\delta; \lambda; z) + H_n^{(2)}(\delta; \lambda; z).$$

$$H_n^{(2)}(\delta; \lambda; z) = - \frac{1}{n^{k/2}} \int_{\{||t|| > \delta \sqrt{n}\}} \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}; n\right)$$

$$\log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}; n\right) dt =$$

$$= - \int_{\{||t|| > \delta \sqrt{n}\}} \frac{v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right)}{c_n} \log\left(n^{k/2} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) / c_n\right) dt =$$

$$= \frac{k \log n}{2c_n} \int_{\{||t|| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt + \frac{\log c_n}{c_n} \int_{\{||t|| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

$$- \frac{1}{c_n} \int_{\{||t|| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log v_n(t) dt -$$

$$- \frac{1}{c_n} \int_{\{||t|| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt ;$$

Define by $J_n(\delta; \lambda; z)$ and $I_n(\delta; \lambda; z)$ the following random variables :

$$(33) \quad J_n(\delta; \lambda; z) = - \int_{\{||t|| > \delta \sqrt{n}\}} v_n(t) \log v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

$$(34) \quad I_n(\delta; \lambda; z) = - \int_{\{\|t\| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

Then,

$$\begin{aligned} J_n(\delta; \lambda; z) &= - \int_{\{\|t\| > \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log v_n(t) dt = \\ &= 2 \int_{\{\|t\| > \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \left(-\sqrt{v_n(t)} \log \sqrt{v_n(t)} \right) dt \end{aligned}$$

Since for every $t \in R^k$, $0 \leq v_n(t) \leq 1$, we obtain,

$$0 \leq -\sqrt{v_n(t)} \log \sqrt{v_n(t)} \leq \frac{1}{e}, \quad (\forall) t \in R^k.$$

Therefore,

$$\begin{aligned} 0 \leq J_n(\delta; \lambda; z) &\leq \frac{2}{e} \int_{\{\|t\| > \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt \leq \\ &\frac{2}{e} \exp(-n\varepsilon(\delta)/2) \int_{\{\|t\| > \delta \sqrt{n}\}} \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt = \\ &\frac{2}{e} n^{k/2} \exp\left(-n \frac{\varepsilon(\delta)}{2}\right) \int_{\{\|\theta - \hat{\theta}_n\| > \delta\}} \lambda(\theta) d\theta \leq \frac{2}{e} K n^{k/2} \exp\left(-n \frac{\varepsilon(\delta)}{2}\right). \end{aligned}$$

where lemma 4 was used.

Finally, we get :

$$(35) \quad 0 \leq J_n(\delta; \lambda; z) \leq \frac{2}{e} K n^{k/2} \exp\left(-\frac{\varepsilon(\delta)}{2} n\right), \text{ hence for any } \alpha > 0,$$

$$(36) \quad J_n(\delta; \lambda; z) = 0 \quad (n^{-\alpha}).$$

From (34) we obtain :

$$(37) \quad |I_n(\delta; \lambda; z)| \leq \int_{\{\|t\| > \delta \sqrt{n}\}} \left| v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \right| dt \leq$$



$$\left\{ \int_{\{|t|>\delta\sqrt{n}\}} v_n^2(t) \lambda^2 \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log^2 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt \right\}^{1/2} =$$

$$= \left\{ \int_{\{|t|>\delta\sqrt{n}\}} v_n^2(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \left(\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log^2 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \right) dt \right\}^{1/2}$$

If we consider $g(x) = x \log^2 x$, then $|g(x)| \leq \frac{4}{e^2}$ for any $x \in [0, 1]$.

Since for $x > 1$, the function g is monotone increasing and $\log x < b(x-1)$ where b is a real positive number, we obtain that for any $x > 1$, $g(x) < x(x-1)^2$.

Therefore, for any $t \in R^k$ and $z \in X^\infty$ such that $\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \leq 1$,

$$\left| \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log^2 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \right| \leq \frac{4}{e^2}.$$

For any $t \in R^k$ and $z \in X^\infty$, such that $\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) > 1$, since $0 <$

$< \lambda(\theta) < M$ for any $\theta \in \Theta$, we obtain :

$$\left| \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log^2 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \right| \leq b^2 M (M^2 + 1).$$

Finally we get : for any $t \in R^k$ and $z \in X^\infty$,

$$\left| \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log^2 \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \right| \leq \max \left\{ b^2 M (M^2 + 1), \frac{4}{e^2} \right\} = B < \infty.$$

From (37) we obtain :

$$|I_n(\delta; \lambda; z)| \leq B^{1/2} \left\{ \int_{\{|t|>\delta\sqrt{n}\}} v_n^2(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt \right\}^{1/2} \leq$$

$$B^{1/2} \exp(-n \varepsilon(\delta)) \left\{ \int_{\{|t|>\delta\sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt \right\}^{1/2} \leq (KB)^{1/2} \exp(-n \varepsilon(\delta))$$

Therefore, for any $\alpha > 0$,

$$(38) \quad I_n(\delta; \lambda; z) = O(n^{-\alpha}), \text{ a.s.} - P_{\theta_0}.$$

Using (36) and (38), we infer,

$$(39) \quad H_n^{(2)}(\delta; \lambda; z) = O(n^{-\alpha}), \text{ a.s.} - P_{\theta_0}.$$

$$(40) \quad H_n^{(1)}(\delta; \lambda; z) = - \int_{\{\|\hat{\theta} - \hat{\theta}_n\| \leq \delta\}} \lambda(\theta; n) \log \lambda(\theta; n) d\theta =$$

$$\begin{aligned} & \frac{\log c_n}{c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt - \frac{k \log n}{2c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt - \\ & - \frac{1}{c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log v_n(t) dt - \\ & - \frac{1}{c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt. \end{aligned}$$

We denote by $A_n(\delta; \lambda; z)$, $B_n(\delta; \lambda; z)$, $D_n(\delta; \lambda; z)$, $W_n(\delta; \lambda; z)$, the following random variables :

$$(41) \quad A_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

$$(42) \quad D_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

$$(43) \quad B_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) \log v_n(t) dt$$

$$(44) \quad W_n(\delta; \lambda; z) = \frac{k \log n}{2c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda\left(\hat{\theta}_n + \frac{t}{\sqrt{n}}\right) dt$$

We also define $E(\delta; \lambda; z)$ by :

$$(45) \quad E_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) H(t, n) dt,$$

where $H(t, n)$ is given by (6).

Since $\left| \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \right| \leq \max \left\{ \frac{1}{e}, M(M-1) \right\} = R < \infty$,
and

$H(t, n) \leq \exp(-tWt') + \exp\left(\frac{1}{2} tA(\theta_0)t'\right)$ for any $t \in R^k$ such that $\|t\| \leq \delta \sqrt{n}$, we conclude that

$\chi_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) H(t, n)$ is dominated by an integrable function.

Since $\lim_{n \rightarrow \infty} H(t, n) = 0$, a.s. $- P_{\theta_0}$, for any t such that $\|t\| \leq \delta \sqrt{n}$, using the bounded convergence theorem, we obtain :

$$(46) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} E_n(\delta; \lambda; z) = 0) = 1$$

From (46), we obviously obtain :

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\{ \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \nu_n(t) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt - \right. \\ & \left. \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \right. \\ & \left. \Phi(-A^{-1}(\theta_0), t) dt \right\} = 0, \text{ a.s. } - P_{\theta_0}. \end{aligned}$$

Since,

$$\chi_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \Phi(-A^{-1}(\theta_0), t)$$

is dominated by the integrable function $R \exp \left(\frac{1}{2} t A(\theta_0) t' \right)$, and since the estimator $\hat{\theta}_n$ is consistent, and λ a continuous function, using again the bounded convergence theorem, we get :

$$(47) \lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \\ \Phi(-A^{-1}(\theta_0), t) dt = \\ = \lambda(\theta_0) \log \lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2}, \text{ a.s. } - P_{\theta_0}.$$

Therefore, we get :

$$\lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) v_n(t) \log \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = \\ = (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \lambda(\theta_0) \log \lambda(\theta_0), \text{ a.s. } - P_{\theta_0}.$$

Thus,

$$(48) P_{\theta_0}(\lim_{n \rightarrow \infty} D_n(\delta; \lambda; z) = (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \lambda(\theta_0) \log \lambda(\theta_0)) = 1.$$

$$(49) B_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log v_n(t) dt = \\ = 2 \int_{\{\|t\| \leq \delta \sqrt{n}\}} (\sqrt{v_n(t)} \log \sqrt{v_n(t)}) \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt.$$

Define $F_n(\delta; \lambda; z)$ by :

$$(50) F_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) |\sqrt{v_n(t)} \log \sqrt{v_n(t)} - \\ - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} \cdot \\ \cdot \log (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)}| dt.$$

Obviously we can obtain for $F_n(\delta; \lambda; z)$ the following simpler expression :

$$(51) F_n(\delta; \lambda; z) = \int_{\{\|t\| \leq \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right)$$

$$\left| \sqrt{v_n(t)} \log \sqrt{v_n(t)} - \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \log \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \right| dt$$

Since $A(\theta_0)$ is negative definite,

$$0 < \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \leq 1 \quad \text{for any } t \in R^k.$$

Therefore, we get :

$$(52) \quad \left| \sqrt{v_n(t)} \log \sqrt{v_n(t)} - \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \log \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \right| \leq \\ \leq |\sqrt{v_n(t)} \log \sqrt{v_n(t)}| + \left| \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \log \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \right| \leq \frac{2}{e}.$$

Using lemma 3, we obtain :

$$(53) \quad \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} \leq M \exp \left(-\frac{1}{2} t W t' \right).$$

From (52) and (53) it is easy to prove that

$$\sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) |\sqrt{v_n(t)} \log \sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \\ \sqrt{\Phi(-A^{-1}(\theta_0), t)} \log (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)}|$$

is dominated by the integrable function $\frac{2}{e} M \exp \left(-\frac{1}{2} t W t' \right)$.

Since for any $t \in R^k$ such that $\|t\| \leq \delta \sqrt{n}$,

$$\lim_{n \rightarrow \infty} |v_n(t - (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \Phi(-A^{-1}(\theta_0), t)| = 0, \text{ a.s. } - P_{\theta_0},$$

using the bounded convergence theorem, we obtain :

$$(54) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} F_n(\delta; \lambda; z) = 0) = 1.$$

From (54) we deduce :

$$(55) \quad \lim_{n \rightarrow \infty} \left\{ \int_{\{\|t\| \leq \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \sqrt{v_n(t)} \log \sqrt{v_n(t)} dt - \right.$$

$$- \int_{\{\|t\| \leq \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \log \exp \left(\frac{1}{4} t A(\theta_0) t' \right) dt \Big\} = 0,$$

a.s. — P_{θ_0} .

Let the random variable $G_n(\delta; \lambda; z)$ be defined by :

$$(56) \quad G_n(\delta; \lambda; z) = \frac{1}{4} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) t A(\theta_0) t' \exp \left(\frac{1}{4} t A(\theta_0) t' \right)$$

$$| \sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} | dt$$

Obviously,

$$(57) \quad \lim_{n \rightarrow \infty} | \sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} | = 0.$$

a.s. — P_{θ_0} , for any $t \in R^k$ such that $\|t\| \leq \delta \sqrt{n}$.

But,

$$(58) \quad | \sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} | =$$

$$= \left| \sqrt{v_n(t)} - \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \right| \leq \sqrt{v_n(t)} + \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \leq$$

$$\leq \exp \left(-\frac{1}{2} t W t' \right) + \exp \left(\frac{1}{4} t A(\theta_0) t' \right),$$

for any $t \in R^k$, such that $\|t\| \leq \delta \sqrt{n}$.

We therefore get ;

$$(59) \quad \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}} t A(\theta_0) t' \exp \left(\frac{1}{4} t A(\theta_0) t' \right)$$

$$| \sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} | \leq$$

$$\leq M t A(\theta_0) t' \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \left(\exp \left(-\frac{1}{2} t W t' \right) + \exp \left(\frac{1}{4} t A(\theta_0) t' \right) \right) =$$

$$= M t A(\theta_0) t' \left(\exp \left(\frac{1}{4} t A(\theta_0) t' - \frac{1}{2} t W t' \right) + \exp \left(\frac{1}{2} t A(\theta_0) t' \right) \right).$$

Since W can be taken as $-\frac{1}{2}(A(\theta_0) + \varepsilon_0 I)$ with $\varepsilon_0 > 0$, we get :

$$\frac{1}{4} tA(\theta_0) t' - \frac{1}{2} tWt' = \frac{1}{2} t(A(\theta_0) + \varepsilon_0 I) t' - \frac{1}{4} \varepsilon_0 \|t\|^2 < 0,$$

for any $t \in R^k$, $\|t\| \neq 0$.

Therefore,

$$\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) \chi_{\{\|t\| \leq \delta \sqrt{n}\}}$$

$\sqrt{v_n(t)} - (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)}$ is dominated by an integrable function.

Using the bounded convergence theorem, we obtain; taking account of (57),

$$(60) \quad P_{\theta_0}(\lim_{n \rightarrow \infty} G_n(\delta; \lambda; z) = 0) = 1.$$

From (60) we infer :

$$(61) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} \left\{ \frac{1}{4} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) \right. \right.$$

$$\left. \sqrt{v_n(t)} dt - \frac{1}{4} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) (2\pi)^{k/4} \right.$$

$$\left. \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} dt \right\} = 0) = 1.$$

Obviously,

$$\lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \chi_{\{\|t\| \leq \delta \sqrt{n}\}}$$

$\sqrt{\Phi(-A^{-1}(\theta_0), t)}$ is dominated by the integrable function

$$MtA(\theta_0) t' \exp \left(\frac{1}{2} tA(\theta_0) t' \right).$$

Using the consistency of the estimator $\hat{\theta}_n$, the continuity of λ and the bounded convergence theorem, we obtain :

$$(62) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) \right. \\ \left. (2\pi)^{k/4} \det(-A^{-1}(\theta_0))^{1/4} \sqrt{\Phi(-A^{-1}(\theta_0), t)} dt = \right. \\ \left. = \lambda(\theta_0) \int tA(\theta_0) t' \exp \left(\frac{1}{2} tA(\theta_0) t' \right) dt \right) = 1.$$

From (61) and (62) we get :

$$(63) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} \frac{1}{4} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \sqrt{v_n(t)} \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) tA(\theta_0) t' \exp \left(\frac{1}{4} tA(\theta_0) t' \right) dt = \right. \\ \left. = \frac{1}{4} \lambda(\theta_0) \int tA(\theta_0) t' \exp \left(\frac{1}{2} tA(\theta_0) t' \right) dt \right) = 1.$$

From (55) and (63) we finally get :

$$(64) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} v_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) \log \sqrt{v_n(t)} dt = \right. \\ \left. = \frac{1}{4} \lambda(\theta_0) \int tA(\theta_0) t' \exp \left(\frac{1}{2} tA(\theta_0) t' \right) dt \right) = 1,$$

therefore,

$$(65) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} B_n(\delta; \lambda; z) = \frac{1}{2} \lambda(\theta_0) \int tA(\theta_0) t' \exp \left(\frac{1}{2} tA(\theta_0) t' \right) dt \right) = 1.$$

Since the matrix $-A(\theta_0)$ is positive definite and symmetric, the eigenvalues of $-A(\theta_0)$ are real positive numbers. Let denote by $\lambda_1, \lambda_2, \dots, \lambda_k$ the eigenvalues of the matrix $-A(\theta_0)$. As it is well known, there exists an orthogonal matrix T , such that $\det(T) \neq 0$ and

$$T(-A(\theta_0))T' = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{bmatrix}.$$

Let denote by I the following integral :

$$I = \int t A(\theta_0) t' \exp \left(\frac{1}{2} t A(\theta_0) t' \right) dt$$

If we consider the transformation of variable $u = tT^{-1}$, we obtain :

$$\begin{aligned} (66) \quad I &= \int u T A(\theta_0) T' u' \exp \left(\frac{1}{2} u T A(\theta_0) T' u' \right) du = \\ &= - \int u (T(-A(\theta_0)) T') u' \exp \left(-\frac{1}{2} u (T(-A(\theta_0)) T') u \right) du = \\ &= - \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \sum_{i=1}^k u_i^2 \lambda_i \exp \left(- \sum_{j=1}^k u_j^2 \lambda_j \right) du_1 \dots du_k = \\ &= - \sum_{i=1}^k \lambda_i \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} u_i^2 \exp \left(- \sum_{j=1}^k u_j^2 \lambda_j \right) du_1 \dots du_k = \\ &= - \sum_{i=1}^k \lambda_i \left(\prod_{\substack{j=1 \\ j \neq i}}^k \int_{-\infty}^{+\infty} \exp(-u_j^2 \lambda_j) du_j \right) \left(\int_{-\infty}^{+\infty} u_i^2 \exp(-u_i^2 \lambda_i) du_i \right). \end{aligned}$$

But,

$$\prod_{\substack{j=1 \\ j \neq i}}^k \left(\int_{-\infty}^{+\infty} \exp(-u_j^2 \lambda_j) du_j \right) = \prod_{\substack{j=1 \\ j \neq i}}^k \frac{1}{\sqrt{\lambda_j}} \int_{-\infty}^{+\infty} \exp(-v_j^2) dv_j = \pi^{(k-1)/2} \frac{\sqrt{\lambda_i}}{\sum_{j=1}^k \sqrt{\lambda_j}}.$$

Also,

$$\int_{-\infty}^{+\infty} u_i^2 \exp(-u_i^2 \lambda_i) du_i = \frac{1}{\lambda_i \sqrt{\lambda_i}} \int_{-\infty}^{+\infty} v_i^2 \exp(-v_i^2) dv_i = \frac{\sqrt{\pi}}{2 \lambda_i \sqrt{\lambda_i}}.$$

Therefore, we obtain :

$$(67) \quad I = \frac{-k (\sqrt{\pi})^k}{2 \prod_{j=1}^k \sqrt{\lambda_j}}$$

hence, finally we get :

$$(68) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} B_n(\delta; \lambda; z) = - \frac{\lambda(\theta_0) k (\sqrt{\pi})^k}{4 \prod_{j=1}^k \sqrt{\lambda_j}} \right) = 1$$

Since,

$$\int_{\{\|t\| > \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = o(n^{-\alpha}), \text{ for any } \alpha > 0, \text{ we obtain :}$$

$$(69) \quad c_n - \int_{\{\|t\| \leq \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = \int_{\{\|t\| > \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = o(n^{-\alpha}), \text{ a.s.} - P_{\theta_0}, \text{ for any } \alpha > 0.$$

Using (11) and (69), we obtain :

$$(70) \quad \lim_{n \rightarrow \infty} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = \lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2},$$

a.s. - P_{θ_0} .

Therefore,

$$(71) \quad P_{\theta_0} (\lim_{n \rightarrow \infty} A_n(\delta; \lambda; z) = \lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2}) = 1.$$

$$\begin{aligned} W_n(\delta; \lambda; z) &= \frac{k \log n}{2c_n} \int_{\{\|t\| \leq \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt = \\ &= \frac{k \log n}{2c_n} \left(c_n - \int_{\{\|t\| > \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt \right) = \\ &= \frac{k}{2} \log n - \frac{k \log n}{2c_n} \int_{\{\|t\| > \delta \sqrt{n}\}} \nu_n(t) \lambda \left(\hat{\theta}_n + \frac{t}{\sqrt{n}} \right) dt. \end{aligned}$$

Hence, for any $\alpha > 0$.

$$(72) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} (W_n(\delta; \lambda; z) - \frac{k}{2} \log n) n^\alpha = 0 \right) = 1.$$

We deduce :

$$(73) \quad H_n(\lambda; z) = H_n^{(2)}(\delta; \lambda; z) + \frac{\log c_n}{c_n} A_n(\delta; \lambda; z) - \\ - W_n(\delta; \lambda; z) - \frac{1}{c_n} B_n(\delta; \lambda; z) - \frac{1}{c_n} D_n(\delta; \lambda; z).$$

Therefore, for any $\alpha > 0$,

$$(74) \quad P_{\theta_0} \left(\lim_{n \rightarrow \infty} n^\alpha (H_n(\lambda; z) + \frac{k}{2} \log n - \frac{\log c_n}{c_n} A_n(\delta; \lambda; z) + \right. \\ \left. + \frac{1}{c_n} B_n(\delta; \lambda; z) + \frac{1}{c_n} D_n(\delta; \lambda; z)) = 0 \right) = 1$$

It is very easy to show that :

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{c_n} B_n(\delta; \lambda; z) + \frac{1}{c_n} D_n(\delta; \lambda; z) - \frac{1}{c_n} \log c_n A_n(\delta; \lambda; z) \right\} = \\ - \frac{\lambda(\theta_0) k \pi^{k/2}}{4 \prod_{j=1}^k \sqrt{\lambda_j}} \frac{1}{\lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2}} + \\ + \frac{(2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \lambda(\theta_0) \log \lambda(\theta_0)}{(2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \lambda(\theta_0)} - \\ - \frac{\lambda(\theta_0) (2\pi)^{k/2} \det(-A^{-1}(\theta_0))^{1/2} \log (2\pi)^{k/2} \lambda(\theta_0) \det(-A^{-1}(\theta_0))^{1/2}}{(2\pi)^{k/2} \lambda(\theta_0) \det(-A^{-1}(\theta_0))^{1/2}}.$$

Since $\det(-A(\theta_0)) = \prod_{j=1}^k \lambda_j$, we get $\det(-A^{-1}(\theta_0))^{1/2} = \frac{1}{\prod_{j=1}^k \sqrt{\lambda_j}}$.

Finally we obtain :

$$(75) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{c_n} B_n(\delta; \lambda; z) + \frac{1}{c_n} D_n(\delta; \lambda; z) - \frac{\log c_n}{c_n} A_n(\delta; \lambda; z) \right) = \\ - \frac{k}{2^{k/2+2}} - \frac{k}{2} \log(2\pi) + \frac{1}{2} \sum_{j=1}^k \log \lambda_j \text{ a.s. } - P_{\theta_0}.$$

Therefore,

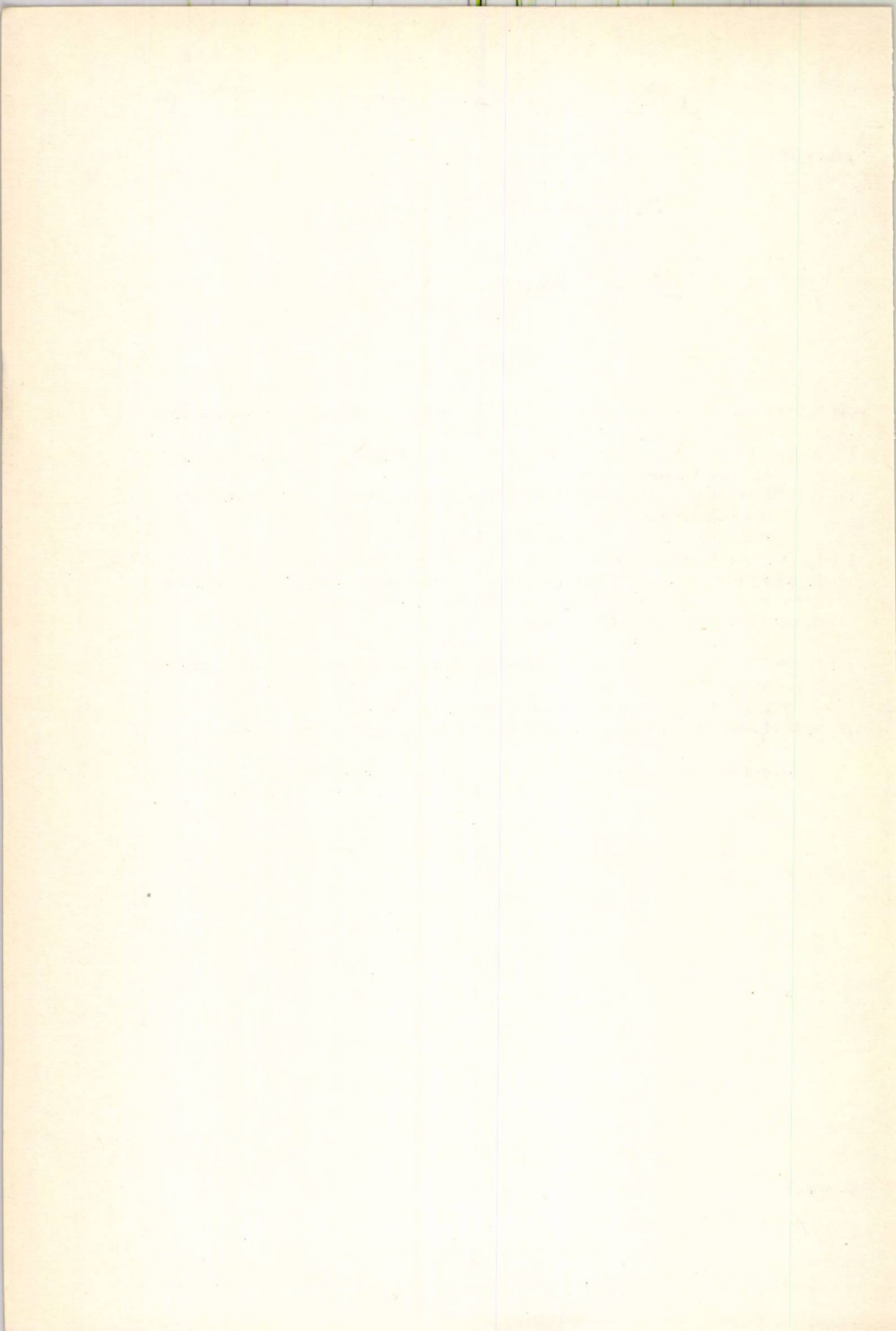
$$\begin{aligned}
 (76) \quad & P_{\theta_0} \left(\lim_{n \rightarrow \infty} (H_n(\lambda; z) + \frac{k}{2} \log n) = \right. \\
 & \left. = \frac{k}{2^{k/2+2}} + \frac{k}{2} \log(2\pi) - \frac{1}{2} \sum_{j=1}^k \log \lambda_j \right) = 1.
 \end{aligned}$$

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LES ESPACES DE RIEMANN AYANT LA FORME DE RICCI À COURBURE CONSTANTE

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On démontre certaines propriétés des espaces de Riemann V_n ayant la forme de Ricci (définie positive et non singulière) à courbure constante. Entre autres, on donne la condition nécessaire et suffisante pour que l'espace d'Einstein V_n soit un espace à courbure constante; c'est donc la forme de Ricci d'avoir la courbure constante qui en constitue la condition. On détermine aussi la forme des métriques des espaces V_n qui sont conformes à un espace euclidien telle que V_n ont la forme de Ricci à courbure constante.

1. Introduction

Soit M une variété différentiable réelle à n dimensions*. Désignons par $\mathcal{F}(M)$ l'anneau des fonctions réelles, différentiables, définies sur M et par $\mathcal{T}^{r,s}(M)$ le $\mathcal{F}(M)$ -module des champs de tenseurs de type (r, s) sur M . En particulier, pour $\mathcal{T}^{1,0}(M)$, nous utiliserons aussi la notation $\mathcal{X}(M)$.

Soit ∇ et $\overline{\nabla}$ deux connexions linéaires sur M et soit $D \in T^{1,2}(M)$ le champ tensoriel de déformation affine [1], c'est-à-dire

$$(1.1) \quad D(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \quad (\forall) X, Y \in \mathcal{X}(M).$$

En définissant le produit de deux champs de vecteurs $X, Y \in \mathcal{X}(M)$ comme étant le champ

$$(1.2) \quad X \circ Y = D(X, Y),$$

nous pouvons légèrement vérifier que le module $\mathcal{X}(M)$ devient une algèbre sur l'anneau $\mathcal{F}(M)$. Cette algèbre est appelée [2] l'algèbre de déformation du paire $(\nabla, \overline{\nabla})$ et elle est notée par $\mathcal{U}(M, \nabla, \overline{\nabla})$.

Soit $K \in \mathcal{T}^{1,3}(M)$ le champ tensoriel de la courbure de déformation, définie par

$$(1.3) \quad K(X, Y)Z = \frac{1}{2} [D(X, D(Y, Z)) - D(Y, D(X, Z))], \\ (\forall) X, Y, Z \in \mathcal{X}(M).$$

* Les variétés différentiables, les applications différentiables, les champs tensoriels et les connexions linéaires qu'interviennent dans cette étude sont supposés de classe C^∞ .

On sait que l'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$ est commutative et associative si, et seulement si, nous avons [2]

$$(1.3') \quad D(X, Y) = D(Y, X) \text{ et } K(X, Y)Z = 0, (\forall) X, Y, Z \in \mathcal{X}(M).$$

Un élément $X \in \mathcal{U}(M, \nabla, \overline{\nabla})$ ayant la propriété que $X_p \neq 0$, $(\forall) p \in M$ est nommé *champ caractéristique* [3], s'il existe une fonction $f \in \mathcal{F}(M)$ de telle sorte que

$$(1.4) \quad D(X, X) = fX.$$

On voit immédiatement que la condition (1.4) est équivalente à

$$(1.4') \quad D(X, X) \otimes X - X \otimes D(X, X) = 0.$$

L'interprétation géométrique des champs caractéristiques de l'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$ se trouve dans [3].

Les trajectoires d'un champ caractéristique s'appellent *les courbes caractéristiques*. Si nous notons par D_{jk}^i les composantes du D dans un système de coordonnées locales $(x^1, \dots, x^n)$, alors, en utilisant la formule (1.4'), nous obtenons le système différentiel des équations des courbes caractéristiques de l'algèbre $\mathcal{U}(M, \nabla, \overline{\nabla})$:

$$(1.5) \quad (D_{ks}^i \delta_r^s - D_{ks}^j \delta_r^j) \frac{dx^k}{dt} \frac{dx^s}{dt} \frac{dx^r}{dt} = 0.$$

Dans le cas $n = 2$ (quand la variété est bidimensionnelle), si l'une des deux connexions linéaires $\nabla, \overline{\nabla}$ est localement euclidienne, on obtient les résultats donnés dans [4].

2. Considérations sur la forme de Ricci

En convenant de faire usage des notations données dans [5], soit V_n un espace de Riemann dont la métrique est

$$(2.1) \quad ds^2 = g_{ij} dx^i dx^j.$$

Notons par $\left| \begin{smallmatrix} i \\ j \quad k \end{smallmatrix} \right|$ les symboles de Cristoffel de seconde espèce de la métrique (2.1) et par R_{jk} les composantes du tenseur de Ricci.

Supposons que la forme de Ricci

$$(2.2) \quad \Phi = R_{ij} dx^i dx^j$$

est définie et positive; de plus, nous considérons qu'elle est non singulière, c'est-à-dire que la forme quadratique (2.2) n'est pas dégénérée.

En ce qui suit, pour éviter les répétitions, nous désignons par $\tilde{V}_n$ un espace de Riemann V_n à métrique (2.1) dont la forme de Ricci (2.2), définie positive et non singulière, est à courbure constante. Alors, les quantités $\begin{vmatrix} i \\ j & k \end{vmatrix}$ sont les symboles de Cristoffel de seconde espèce de la forme (2.2).

Théoreme 2.1. *Si l'espace de Riemann V_n ($n > 1$) est un espace à courbure constante (non nulle), alors la forme de Ricci est aussi à courbure constante.*

En effet, si V_n est un espace à courbure constante, alors les composantes du tenseur de courbure de l'espace V_n s'écrivent sous la forme

$$(2.3) \quad R_{jkl}^i = K(\delta_k^i g_{jl} - \delta_l^i g_{jk}), \quad K = \text{const.}$$

En supposant $i = k$ dans (2.3) et en sommant, nous obtenons

$$(2.3') \quad R_{ji} = (n - 1) K g_{ji};$$

d'où, pour $n > 1$ et $K = 0$, il résulte

$$(2.3'') \quad \begin{vmatrix} i \\ j & k \end{vmatrix} = \begin{vmatrix} i \\ j & k \end{vmatrix}.$$

Donc le tenseur de courbure de la forme de Ricci coïncide avec le tenseur de courbure de l'espace V_n . Par conséquent, en tenant compte des (2.3'), la forme de Ricci est à courbure constante.

En tenant compte des (2.3''), on trouve $D_{jk}^i = 0$; alors il résulte

Proposition 2.1. *Soit $\tilde{V}_n$ un espace de Riemann à courbure constante. L'algèbre de déformation $\mathcal{U}\left(\tilde{V}_n, \begin{vmatrix} i \\ j & k \end{vmatrix}, \begin{vmatrix} i \\ j & k \end{vmatrix}\right)$ est commutative et associative.*

Si l'on considère la forme (1.4), on obtient

$$D(X, X) = 0, \quad (\forall) X \in \mathcal{X}(V_n).$$

Il s'en suit de là que

Proposition 2.2. *Les champs caractéristiques de l'algèbre de déformation $\mathcal{U}\left(V_n, \begin{vmatrix} i \\ j & k \end{vmatrix}, \begin{vmatrix} i \\ j & k \end{vmatrix}\right)$, ou V_n est un espace de Riemann à courbure constante, sont des éléments nilpotents d'indice deux et réciproquement.*

Théorème 2.2. *Tout espace d'Einstein $V_n (n \geq 3)$ est un espace à courbure constante.*

En effet, soit $\tilde{V}_n (n \geq 3)$ un espace d'Einstein; dans ce cas, la forme de Ricci (2.2) est proportionnelle à la métrique de l'espace V_n , c'est-à-dire

$$(2.4) \quad R_{ij} = \rho g_{ij}.$$

Supposons maintenant que nous avons $\rho \neq 0$; alors, en tenant compte des (2.4), nous avons

$$(2.5) \quad \left| \begin{array}{c} i \\ j \quad k \end{array} \right| = \left| \begin{array}{c} i \\ j \quad k \end{array} \right| + \delta_j^i \rho_k + \delta_k^i \rho_j - g_{jk} \rho^i,$$

où

$$(2.5') \quad \rho_i = \frac{\partial \rho}{\partial x^i}, \quad \rho^i = g^{ij} \rho_j.$$

D'autre part, on sait que [6], si $n = 3$, l'espace d'Einstein V_n est à courbure constante; donc, d'après le Théorème 2.1, aussi la forme de Ricci (supposée non singulière) est à courbure constante. On sait aussi que (v., par exemple, [7], I, pag. 292), si $n > 3$, la quantité ρ est une constante: $\rho = \text{const.}$ Donc, pour $n > 3$ et $\rho = \text{const.}$ ($\rho \neq 0$), les formules (2.5) sont réduites aux expressions (2.3''), de telle sorte que

$$\bar{R}_{kl}^i = \bar{K}(\delta_k^i R_{jl} - \delta_l^i R_{jk}) = K(\delta_k^i g_{jl} - \delta_l^i g_{jk}) = R_{kl}^i,$$

où $K = \bar{K} \rho$, ce qui montre que l'espace $\tilde{V}_n$ est à courbure constante.

Remarque. Les Théorèmes 2.1 et 2.2 expriment:

Théorème 2.3. *Une condition nécessaire et suffisante pour qu'un espace d'Einstein $V_n (\rho \neq 0)$ soit un espace à courbure constante est que la forme de Ricci soit à courbure constante.*

Corollaire. *L'algèbre de déformation $\mathcal{U} \left(\tilde{V}_n, \left| \begin{array}{c} i \\ j \quad k \end{array} \right|, \left| \begin{array}{c} i \\ j \quad k \end{array} \right| \right)$ d'un espace d'Einstein $\tilde{V}_n (n \geq 3)$ est commutative et associative, les champs caractéristiques de l'algèbre étant les seuls éléments nilpotents d'indice deux.*

Si $n = 2$, on peut toujours avoir

$$(2.6) \quad R_{ij} = K g_{ij}, \quad K = \frac{R}{2},$$

où K est la courbure riemannienne et R est la courbure scalaire de l'espace V_2 .

Remarque. La forme de Ricci est non singulière si, et seulement si, on a $K \neq 0$, ce qui nous supposons.

En introduisant les (2.6) dans (2.2), les symboles de Christoffel de seconde espèce de la forme (2.2) s'écrivent

$$(2.7) \quad \overline{\begin{vmatrix} i \\ j & k \end{vmatrix}} = \begin{vmatrix} i \\ j & k \end{vmatrix} + \frac{1}{2K} (\delta_j^i R_k + \delta_k^i R_j - g_{jk} g^{is} R_s) \text{ où } R_i = \frac{\partial R}{\partial x^i}.$$

Si $K = \text{const.}$, l'espace V_2 est à courbure constante; dans ce cas, on a

$$\overline{\begin{vmatrix} i \\ j & k \end{vmatrix}} = \begin{vmatrix} i \\ j & k \end{vmatrix};$$

donc il résulte que la forme de Ricci est à courbure constante (v. le Théorème 2.1).

Si $K \neq \text{const.}$, les formules (2.7) nous montre que les composantes du tenseur de déformation affine sont

$$(2.8) \quad D_{jk}^i = \frac{1}{2K} (\delta_j^i R_k + \delta_k^i R_j - g_{jk} g^{is} R_s).$$

En tenant compte des (2.8) dans le système qui donne les directions caractéristiques

$$(2.9) \quad (D_{jk}^i \delta_r^p - D_{jk}^p \delta_r^i) X^j X^k X^r = 0,$$

nous trouvons

$$(2.9') \quad g_{jk} X^j X^k (g^{is} R_s X^p - g^{ps} R_s X^i) = 0.$$

Ces formules nous donnent

$$X^i = \lambda g^{is} R_s$$

Par conséquent, dans chaque point nous avons la direction caractéristique

$$(X^1, X^2) = \left(g^{1s} \frac{\partial K}{\partial x^s}, g^{2s} \frac{\partial K}{\partial x^s} \right).$$

Nous avons ainsi

Proposition 2.3. L'algèbre de déformation $\mathcal{U}\left(V_2, \begin{vmatrix} i \\ j & k \end{vmatrix}, \overline{\begin{vmatrix} i \\ j & k \end{vmatrix}}\right)$, associée à un espace de Riemann V_2 ayant la courbure riemannienne K différente d'une constante, admet un champ caractéristique.

En ce qui concerne la courbure de déformation, elle a les composantes

$$(2.10) \quad K_{jkl}^i = \delta_k^i (\sigma_j \sigma_l - g_{jl} \sigma_s \sigma^s) - \delta_l^i (\sigma_j \sigma_k - g_{jk} \sigma_s \sigma^s) + \\ + (g_{jl} \sigma_k - g_{jk} \sigma_l) \sigma^i,$$

où

$$(2.10') \quad \sigma_i = \frac{1}{2K} R_i.$$

Nous avons donc

Proposition 2.4. L'algèbre de déformation $\mathcal{U} \left(V_2, \begin{vmatrix} i & \\ j & k \end{vmatrix}, \overline{\begin{vmatrix} i & \\ j & k \end{vmatrix}} \right)$ est commutative et, en général, n'est pas associative.

Par exemple, on sait que ([8], III, p. 385), étant données deux formes quadratiques

$$(2.11) \quad e^{a-f} (du^2 + dv^2), \quad e^a (du^2 + dv^2),$$

elle sont la première forme et respectivement la forme de Ricci d'une surface ayant la courbure gaussienne $K = e^f$, si les fonctions a et f sont liées par la relation

$$\Delta a + 2e^a = \Delta f \left(\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right).$$

Parce que la courbure gaussienne de la forme de Ricci est

$$\bar{K} = -\frac{1}{2} e^{-a} \Delta a,$$

elle est constante (et égale à 1) si, et seulement si, on a

$$\Delta a + 2e^a = 0.$$

Dans ce cas, il résulte que nous avons $K_{jkl}^i = 0$.

On peut donc énoncer :

Proposition 2.5. L'algèbre de déformation associée aux formes (2.11) est commutative et associative, tandis que la direction

$$(2.12) \quad (X^1, X^2) = \left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right)$$

est une direction caractéristique.

§3. Métriques des espaces ayant la forme de Ricci à courbure constante

Cherchons maintenant s'il est possible de trouver des espace de Riemann $\tilde{V}_n$ dont la métrique soit de la forme

$$(3.1) \quad ds^2 = e^{2\varphi} c_{ij} dx^i dx^j,$$

où φ est une fonction des variables $x^1, \dots, x^n$ et c_{ij} sont des constantes satisfaisant les conditions $c_{ij} = c_{ji}$, $|c_{ij}| \neq 0$.

Les symboles de Cristoffel de seconde espèce de la métrique (3.1) sont donnés par

$$(3.2) \quad \left| \begin{smallmatrix} i \\ j \quad k \end{smallmatrix} \right| = \delta_j^i \varphi_k + \delta_k^i \varphi_j - \delta_{jk} c_{is} \varphi_s \text{ où } \varphi_i = \frac{\partial \varphi}{\partial x^i}.$$

Le tenseur de courbure de la connexion (3.2) a les composantes

$$(3.3) \quad R_{jkl}^i = c_{jk} c^{iq} (\varphi_{lq} - \varphi_l \varphi_q) - \delta_k^i (\varphi_{lj} - \varphi_l \varphi_j) - \\ - c_{jl} c^{iq} (\varphi_{kq} - \varphi_k \varphi_q) + \delta_l^i (\varphi_{kj} - \varphi_k \varphi_j) + (\delta_l^i c_{jk} - \delta_k^i c_{jl}) c^{pq} \varphi_p \varphi_q$$

et le tenseur contracté de courbure (le tenseur de Ricci) est donné par

$$(3.4) \quad R_{jl} = - (n - 2) (\varphi_{jl} - \varphi_j \varphi_l) - c_{jl} c^{pq} \varphi_p \varphi_q - (n - 2) c_{ji} c^{pq} \varphi_p \varphi_q,$$

où

$$(3.5) \quad \varphi_{jk} = \frac{\partial^2 \varphi}{\partial x^j \partial x^k}.$$

Cas I : Supposons maintenant que tous les composantes du tenseur contracté de courbure sont des constantes.

Dans ce cas, on dit que la forme de Ricci (2.2) est une forme euclidienne.

Soit

$$c^{jl} R_{jl} = - (n - 1) c^{jl} [2 \varphi_{jl} + (n - 2) \varphi_j \varphi_l].$$

En différentiant cette relation, nous obtenons

$$c^{jl} [2 \varphi_{jlk} + (n - 2) (\varphi_{jk} \varphi_l + \varphi_j \varphi_k)] = 0,$$

c'est-à-dire

$$\varphi_{jlk} = - \frac{n - 2}{2} (\varphi_{jk} \varphi_l + \varphi_j \varphi_{lk}).$$

En tenant compte de ces formules dans $\frac{\partial R_{ll}}{\partial x^k} = 0$, nous avons

$$-n(\varphi_{jk}\varphi_l + \varphi_j\varphi_{lk}) + c_{jl}c^{pq}(\varphi_{pk}\varphi_q + \varphi_p\varphi_{qk}) = 0,$$

ou bien

$$-nc^{jr}(\varphi_{jk}\varphi_l + \varphi_j\varphi_{lk}) + \delta_l^r c^{pq}(\varphi_{pk}\varphi_q + \varphi_p\varphi_{qk}) = 0.$$

Pour $l \neq r$, cette formule donne

$$\varphi_{jk}\varphi_l + \varphi_j\varphi_{lk} = 0, \text{ ou bien } \frac{\partial}{\partial x^k}(\varphi_j\varphi_l) = 0;$$

d'où il résulte que

$$(3.6) \quad \varphi = a_i x^i,$$

a_i étant des constantes.

En tenant compte maintenant de (3.6) dans la métrique (3.1) de V_n , nous obtenons la forme donnée par V. Wagner [9]

$$(3.7) \quad ds^2 = e^{2a_i x^i} c_{jk} dx^j dx^k.$$

D'où le résultat suivant:

Théorème 3.1. Tout espace de Riemann V_n à métrique (3.1), $n > 2$, ayant la forme de Ricci euclidienne est un espace de Wagner.

Cas II : Considérons maintenant le cas où l'espace $\tilde{V}_n$ est un espace d'Einstein et $c_{ij} = \delta_{ij}$, donc

$$(3.8) \quad R_{jk} = \frac{R}{n} e^{2\varphi} \delta_{jk},$$

R étant l'invariant scalaire de courbure de l'espace $\tilde{V}_n$. Alors

$$(3.9) \quad R_{jkl}^i = \lambda e^{2\varphi} (\delta_k^i \delta_{lj} - \delta_l^i \delta_{jk})$$

où $\lambda = \frac{R\bar{K}}{n}$, $\bar{K}$ étant la courbure riemannienne de la forme de Ricci.

En faisant la notation

$$(3.10) \quad \psi_{jk} = \varphi_{jk} - \varphi_j \varphi_k + \frac{1}{2} \delta_k \sum_s \varphi_s^2$$

et en tenant compte des (3.3) où $c_{ij} = \delta_{ij}$ dans les conditions (3.9), nous obtenons

$$(3.9') \quad \delta_{jk} \psi_{il} - \delta_k^i \psi_{jl} - \delta_{jl} \psi_{ki} + \delta_l^i \psi_{jk} = \lambda e^{2\varphi} (\delta_k^i \delta_{jl} - \delta_l^i \delta_{jk}).$$

Les relations (3.9'), pour $i = k \neq j \neq l$ et $i = k \neq j = l$, nous donnent

$$(3.9'') \quad \psi_{jl} = 0 \ (j \neq l), \quad -\psi_{jj} - \psi_{ii} = \lambda e^{2\varphi} \ (i \neq j),$$

c'est-à-dire

$$(3.9''') \quad \varphi_{jl} - \varphi_j \varphi_l = 0, \quad \varphi_{ii} + \varphi_{jj} - \varphi_i^2 - \varphi_j^2 + \sum_s \varphi_s^2 + \lambda e^{2\varphi} = 0.$$

En posant

$$(3.10) \quad \varphi = -\ln \rho,$$

les formules (3.9''') s'écrivent

$$(3.11) \quad \frac{\partial^2 \rho}{\partial x^j \partial x^i} = 0, \quad -\rho \left(\frac{\partial^2 \rho}{(\partial x^i)^2} + \frac{\partial^2 \rho}{(\partial x^j)^2} \right) + \sum_s \left(\frac{\partial \rho}{\partial x^s} \right)^2 + \lambda = 0.$$

La première équation (3.11) possède la solution générale

$$(3.10') \quad \rho = \sum_i \rho_i(x^i),$$

où ρ_i est une fonction seulement de la variable x^i . Dans ce cas, la seconde équation (3.11) s'écrit

$$(3.11') \quad -\rho (\rho_i'' + \rho_j'') + \sum_s \rho_s'^2 + \lambda = 0 \ (i \neq j),$$

admettant la solution

$$(3.10'') \quad \rho = a \sum_s (x^s)^2 + b_s x^s + c \ (a, b_s, c - \text{constantes})$$

de telle sorte que

$$(3.12) \quad -4ac + \sum_s b_s^2 + \lambda = 0.$$

Si la constante λ est négative, on peut satisfaire l'équation (3.12), dans le domaine réel et par une transformation orthogonale de variables, en posant

$$(3.13) \quad a = c = 0, \quad b_s = b \delta_s^1.$$

Alors l'espace d'Einstein $\tilde{V}_n$ est un espace à courbure constante négative dont la métrique peut être réduite à la *forme canonique de Beltrami* [10]

$$(3.14) \quad ds^2 = \frac{(dx^1)^2 + \dots + (dx^n)^2}{-\lambda(x^1)^2}.$$

De même, quel qu'il soit la valeur positive, négative ou nulle de la constante λ , on peut satisfaire la condition (3.12) en posant

$$(3.15) \quad b_i = 0, \quad c = 1 \quad (i = 1, 2, \dots, n).$$

Dans ce cas, on a un espace d'Einstein $\tilde{V}_n$ à courbure constante et on peut écrire la métrique de l'espace sous la *forme canonique de Riemann* [11]

$$(3.16) \quad ds^2 = \frac{(dx^1)^2 + \dots + (dx^n)^2}{\left[1 + \frac{\lambda}{4} \sum_s (x^s)^2\right]^2}.$$

On arrive à un résultat compris dans le Théorème 2.2; néanmoins, nous l'énonçons (v. et [6]).

Théorème 3.2. *Tout espace d'Einstein V_n à métrique (3.1) ayant la forme de Ricci à courbure constante est un espace à courbure constante, la métrique de V_n pouvant être réduite à la forme de Beltrami (3.14), quand la courbure λ est négative, ou à la forme de Riemann (3.16), dans le cas général.*

Evidemment, nous pouvons considérer aussi d'autres types de métriques, non seulement la métrique d'un espace V_n conforme à une métrique euclidienne.

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PROF. DR. DOCENT IONEL BUCUR
1930 — 1976

Cadrele didactice și studenții Facultății de matematică, din București, și membrii Societății de științe matematice din R.S.R., anunță cu adîncă durere înțetarea din viață, în ziua de 6 septembrie 1976, a unuia dintre cei mai valoroși matematicienii români, exemplu de înaltă profesionalitate și de dăruire pentru treburile obștești, profesor dr. Ionel Bueur, prorector al Universității din București, și vicepreședinte al Societății de științe matematice din R.S.R. Născut în 1930, și-a luat doctoratul în matematici în 1958, sub conducerea academicianului Simion Stoilow, însușindu-și o vastă cultură în domenii dintre cele mai dificile ale matematicii moderne. Decan al facultății și director general al Institutului central de matematică pînă în clipa îmbolnăvirii sale, Ionel Bueur este autorul unor lucrări de mare finețe în topologie, geometria algebrică, teoria categoriilor logica matematică și în alte domenii. Modul său natural și plin de pasiune de a aborda matematica, contribuțiile sale științifice i-au adus repede un renume internațional. Autor — singur sau în colaborare — al unor monografii publicate de prestigioase edituri românești sau de peste hotare, Ionel Bueur a fost invitat de multe ori de universități străine, pentru cursuri și conferințe. Membru devotat al P.C.R., în cadrul căruia a îndeplinit importante funcții de răspundere, Ionel Bueur a fost răsplătit, pentru meritele sale deosebite, științifice și obștești, cu ordine și medalii ale Republicii Socialiste România. Prof. Ionel Bueur a fost pînă în ultima clipă a vieții sale un spirit generos, de o adîncă umanitate, a cărui imagine rămîne exemplară pentru toți matematicienii români și pentru toți cei care i-au fost colegi sau studenți.

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Cuvîntul Prof. dr. GEORGE CIUCU — Rectorul Universităţii

A încetat din viaţă în ziua de 6 septembrie 1976, profesorul dr. Ionel Bucur, unul dintre cei mai valoroşi matematicieni români, exemplu de înaltă profesionalitate şi de dăruire pentru treburile obşteşti. Născut la 14 iunie 1930, a absolvit Facultatea de matematică în 1953 şi a trecut doctoratul în matematică în 1958, sub conducerea academicianului Simion Stoilow, însuşindu-şi o vastă cultură în domenii dintre cele mai dificile ale matematicii moderne. Decan al Facultăţii de matematică şi director al Institutului central de Matematică pînă în clipa imbolnăvirii sale, rector al Universităţii din Bucureşti, vicepreşedinte al Societăţii de ştiinţe matematice din R.S.R., Ionel Bucur este autorul unor lucrări de mare fineţe şi profunzime în topologie, geometria algebrică, teoria categoriilor, logica matematică şi în alte domenii. Modul său natural şi plin de pasiune de a aborda matematica, contribuţiile sale ştiinţifice i-au adus repede un renume pe plan internaţional. A fost invitat de multe ori de universităţi străine din Franţa, Italia, Anglia, Canada, R.F.G., Elveţia pentru cursuri şi conferinţe, unde a stabilit contacte ştiinţifice strinse cu o serie de mari matematicieni cum ar fi A. Grothendieck, P. Deligne, P. Hilton, A. Andreotti, B. Eckmann şi alţii. Este autorul primei monografii în limba română de teoria categoriilor şi algebră omologică, iar în colaborare a publicat în limba engleză monografia „Introduction to the Theory of Categories and Functors” (editura John Wiley & Sons) lucrare de referinţă care a fost tradusă şi în limba rusă.

A impresionat pe toţi cei ce l-au cunoscut prin profunzimea lucrărilor sale, prin deosebit de vastă sa cultură matematică, într-un cuvînt prin spiritul său de matematician autentic. Prelegerile sale rămîn un model pentru nivelul lor ridicat, claritatea ideilor, modul ingenios, de aplicare a tehnicilor moderne la importante probleme clasice, precum şi pentru perspectiva ce o deschideau celor ce-l audiau. Creator de şcoală, a îndrumat şi format mulţi tineri matematicieni, oferindu-le cu generozitate tot sprijinul. Ca decan al facultăţii în perioada 1972—1976 a coordonat cu competenţă, tact şi clarviziune toate sarcinile cu care facultatea a fost confruntată. Membru devotat al P.C.R., în cadrul căruia a îndeplinit importante funcţii de răspundere, Ionel Bucur a fost răsplătit, pentru meritele sale ştiinţifice şi obşteşti, cu ordine şi medalii ale Republicii Socialiste România, Prof. Ionel Bucur a fost pînă în ultima clipă a vieţii sale un spirit ales, de o adîncă umanitate a cărei imagine rămîne exemplară pentru toţi matematicienii români şi pentru toţi cei care i-au fost colegi sau studenţi.

Disparaţiia sa prematură, în plină putere creatoare ne-a îndurerat profund pe noi toţi şi reprezintă o pierdere irecuperabilă pentru matematică.

Cuvîntul Prof. dr. doc. CABIRIA ANDREIAN

Pătrunşi de adîncă durere ne luăm rămas bun de la Ionel Bucur, de la colegul şi prietenul drag, de la profesorul şi îndrumătorul generos, de la matematicianul strălucit atît de preţuit în lumea matematică internaţională pentru vasta sa erudiţie, pentru lucrările profund originale, pentru cărţile sale prin care a deschis multora, din ţară şi din străinătate, calea către teoria categoriilor şi algebra omologică.

Crunta realitate ne desparte prea timpuriu de el, răpindu-i pe neaşteptate posibilitatea de a-şi continua activitatea creatoare.

Persistă viu în minţile noastre ultima sa imagine ca Decan al Facultăţii şi Director al Institutului de matematică, plin de înţelepciune, înţelgere şi căldură, cu spiritul său drept, oricînd pregătît să ajute, nelipsit din mijlocul nostru în orice împrejurare importantă.

În aceste clipe dureroase, copleşită de amintiri, desprind o puternică asemănare între Ionel Bucur şi profesorul său, profesorul nostru, Simion Stoilow: aceeaşi distincţie firească,

aceiași fermitate cumpănită și umanism cald, aceiași pasiune pentru orizontul larg matematic, aceiași spirit deschis înnoirilor științei noastre, aceiași interes pentru problemele dificile, aceiași entuziasm cu care organizează și însușește tradiționalele ședințe de comunicări, aceiași dăruire cu care formează tineri matematicieni, aceiași constanță preocupare pentru dezvoltarea matematicii românești, aceiași participare intensă la problemele sociale, aceiași activitate complexă înfăptuită pînă în ultima clipă.

După cum importante rezultate obținute de Ionel Bucur rămîn înscrise în matematica epocii noastre, după cum viața sa închinată cu înflăcărare matematicii și țării noastre rămîne pentru noi un exemplu. zîmbetul său luminos, care ne-a destrămat de atîtea ori îndoile și descurajare, rămîne adînc sădit în inimile noastre.

Cuvîntul Acad. OCTAV ONICESCU

Îndurerată familie, iubiți prieteni și colegi,

Un vâl de rău prevestitoare tristete s-a așternut peste inima și gîndurile noastre la știrea unei grave operații la care Ionel Bucur trebuia supus, tăind astfel brusc firele numeroaselor activități pe care tinărul decan, plin de înțelepciune, le desfășura între zidurile credincioasei cetăți a matematicii românești care este facultatea noastră.

Astăzi, vâlul, acum cernit, îl desparte de noi pentru totdeauna. Dar nu întunecă amintirea vie a inteligenței lui rafinate, a culturii sale încordată spre desăvîrșire, a bunătății sale nesfîrșite, a deschiderii lui bucuroase la toate sugestiile ce puteau aduce un spor de lumină în știința noastră, la întreținerea puterilor de creație a celor veghietori în această cetate, la continuă creștere a capacităților ei științifice și educative.

Sugestiei profesorului Gh. Galbură datorez bucuria de a fi găzduit în Seminarul meu cea dintîi expunere în *Algebră* a tinărului Ionel Bucur.

Vorbind cred, despre clasele Chern ale unei varietăți algebrice, demonstrea o claritate adîncă a ideilor, o sistemă de expunere ca a unui maestru experimental deopotrivă în arta gîndirii și în aceea a comunicării ei, așa cum s-a manifestat, neuitatul nostru tinăr prieten, de-a lungul întregii sale cariere, așa de scurtă, dar așa de strălucită.

La știința lui am apelat cînd am voit să aprofundez interpretarea de spațiu fibrat care împacă mecanica newtoniană cu reprezentare caudridimensională a lui Minkowski-Einstein a mișcării punctului material. Și sigur sînt că sîntem încă departe de a fi scos consecințele matematice și filosofice cele mai importante din modul de prezentare a lui Bucur a acestui spațiu fibrat.

Dar cu aceasta nu este vorba decît de un moment, accidental oarecum, în opera bogată a vieții acestui dăruit matematician care și-a menținut preocupările pe linia culmilor științei noastre.

Această linie ca și demonul interior al gîndirii sale incitată de structurile algebrice l-a dus repede către problemele fundamentelor matematicii și, natural, la acele, mai precise, ale logicii matematice, care l-au angajat adînc în cercetarea rosturilor în cunoaștere și în acțiune ale matematicii zilelor noastre.

Dar și aceste căi deschise ale unei exemplare existențe s-au oprit brusc, fără continuare și fără întoarcere.

Ne rămîne doar datoria de a continua, mai exact să continuați voi tinerii, care l-ați prețuit, l-ați iubit ca și el pe voi, să continuați opera pe aceste drumuri ale neîncelatei perfecționări ce singură ne consolează de surprizele neîndurătoare ale vieții.

Și atunci vom onora, veți onora, împăcați, memoria acestui nobil om care a fost Ionel Bucur.

Cuvîntul Acad. N. TEODORESCU

Durerea vie amușește limba, dar stîrnește gîndul. Puțină pierderii profesorului Ionel Bucur ne doare de multă vreme și durerea tuturor celor ce l-am apropiat este vie de atunci cînd ne-a semănat-o în sufele spectrul ei ce se apropia.

Totuși speranțe luminoase ne alungau îngrijorarea pentru puține zile și ne făceau să simțim că ocupă în ființa noastră un loc de preț pe care îl îndrăgeam.

Pușini sînt cei ce ocupînd locuri de vază în societate nu trezesc invidii și resentimente, păstrînd prietenii avute înainte și poate cîștigînd altele noi. Ionel Bucur era dintre acești pușini prin firea sa blîndă, prin echilibrul senin pe care îl radia prin înțelegerea caldă pe care o avea pentru grijile și necazurile ca și pentru scăderile și slăbiciunile, ca și pentru greșelile și lipsurile celor ce îi erau încredințați pentru a-i conduce.

L-am cunoscut de cînd era student și i-am păstrat totdeauna imaginea aceluia zîmbet timid și stingher care îi lumina figura, dezvăluindu-i firea deschisă și cinstită în care puteam avea încredere desăvîrșită.

L-am urmărit mersul discret înainte în cariera sa de cercetător și profesor, carieră muncită cu o ripă fără pregel și fără ostentație. L-am văzut crescînd și afirmîndu-se în domeniul delicat al algebrei, ca unul din promotorii teoriei categoriilor.

L-am urmărit și în activitatea sa obștească și de conducere unde a reușit să nezească asperități, să risipească nori și să mențină un echilibru care adesea țârea nesigur.

Societatea de științe matematice din R. S. România s-a bucurat de sprijinul său înțelegător și prețios. Ca vice președinte s-a dovedit activ și însușit de dorința de a contribui puternic la ridicarea prestigiului ei, la întărirea rolului ei de societate de masă a tuturor matematicienilor profesioniști ca cercetători sau profesori, precum și a utilizatorilor și beneficiarilor matematicii. Nu putem uita că prin sprijinul său, societatea a obținut două camere care îi erau indispensabile și care i-au întărit dreptul de cetate în localul facultății de matematică. Nu putem uita că deși suprasolicitat ca decan, își găsea timpul necesar pentru a urmări conținutul gazetei matematice, așa cum era propus să fie publicat, dînd sfaturi și asigurîndu-i un nivel din ce în ce mai bun. Nu putem uita activitatea sa în Biroul Societății și ajutorul pe care ni l-a dat întotdeauna pentru a trece peste unele dificultăți care ni se păreau mari.

Pentru toate acestea îi păstrăm și îi vom păstra o caldă recunoștință.

Centrul de Calcul al Universității București s-a bucurat de asemenea de sprijinul său inimos. Deși algebrist terocetician, Ionel Bucur a fost printre primii matematicieni teoreticieni care au înțeles valoarea gnoseologică a informaticii și a luptat pentru a-i menaja un loc demn în învățămîntul din facultatea de matematică pe care o conducea ca decan. Mai mult decît atît, el însuși a abordat cercetări de algebră aplicată informaticii și s-a dovedit încă odată a fi un om de știință deschis aplicațiilor pe care le considera ca surse prețioase de inspirație. Nu vom uita că a înțeles și a sprijinit eficient crearea laboratorului de matematică acordîndu-i spațiul necesar și bunăvoința sa, că a dorit sincer și concret simbioza dintre facultate și Centrul de Calcul dovedind prin aceasta că are o viziune solidă și largă a matematicii revoluției tehnico-științifice.

Pierderea sa ne doare adinec și viu, fiind nedreaptă și profund dăunătoare societății românești, învățămîntului și cercetării matematice din țara noastră. Rugăm crunt lovita familie să ne considere alături în această grea încercare și în sentimentele de prețuire și vie amintire pe care i le va păstra, pe care toți cei ce l-au cunoscut și iubit le vom cultiva în inimile noastre.

Cuvîntul Prof. dr. doc. GH. GALBURĂ

Era într-o sîmbătă de mai cînd ne-am despărțit aici de Ionel Bucur cu surisul său luminos, blind și increzător, bucuros că și-a făcut datoria. A treia zi, am aflat că se operase de urgență. După două săptămîni, în camera sa de la Colțea, era un pelerinaj neîntreput de studenți, colegi mai tineri și mai în vîrstă, dornici să-i arate prietenia și simpatia, urîndu-i grabnică însănătoșire, în care speram cu toții. Și el ne arăta tuturor prietenie caldă și un mare atașament, atașamentul său de viață, pe care a iubit-o în tot ce are ca frumos, interesant și fascinant. Apoi o bănuială teribilă a început să circule tot mai insistent, iar el s-a retras la casa lui, unde părea că va învinge și această grea încercare, cum, datorită bunei sale tovarășe de viață, învinsese și alte mari primejdii, repurtînd apoi însemnate succese în vizitele sale de studii în mari centre universitare. Dar această retragere era să fie pentru el și pentru ai săi un lung martiriu, la al cărui deznodămint primim acum coplesiiți de durere.

Ca eminent student, Ionel Bucur a fost o glorie a facultății noastre între anii 1948—1953.

Cum observa Moisil, Bucur a fost printre primii cercetători la Institutul de Matematică al Academiei R.P.R. formați pe băncile facultății noastre în condițiile grele din primii ani de după război. Și-a început activitatea de cercetare cu probleme de analiză pe varietăți, obținînd titlul de doctor cu o teză remarcabilă asupra unor proprietăți globale ale liniilor geodezice pe un spațiu Finsler. Cu aceasta, și-a atras prețuirea profesorilor facultății și conducerii institutului, devenind în 1963 secretar științific al acestuia.

În 1962—63, Ionel Bucur a fost animatorul principal al unui seminar, în care, timp de doi ani a dezbătut problemele noi ale teoriei categoriilor. Expunerile sale de atunci au stat la baza cărții de algebră omologică și teoria categoriilor din 1964, care a dat o nouă orientare și un puternic avînt cercetărilor de algebră în țara noastră. Cealaltă carte a sa, *Introducere în teoria categoriilor și a functorilor*, scrisă în colaborare cu Deleanu și apărută în limba engleză într-o editură străină, a fost tradusă și în alte limbi. În același timp, el a manifestat o nestăvilită sete de cunoaștere pentru disciplinele centrale ale matematicii: algebra, topologia algebrică, (în lumina ideilor înnoitoare ale lui Leray), teoria grupurilor Lie, geometria algebrică, teoria algoritmilor și a funcțiilor recursive, logica matematică. A publicat o serie de lucrări valoroase privind clasele caracteristice ale unei varietăți algebrice și ale unei varietăți diferențiable. Alte lucrări ale sale se referă la grupuri Lie, puteri Stenrod reduse și clase Chern. A adus contribuții esențiale în teoria descompunerilor în inele subcomutative, obținînd rezultate valoroase într-o problemă importantă a algebrei, abordată mai înainte de Barbilian și reluată ulterior de către elevii lui Bucur.

În scurtele sale popasuri de studii în Franța, Italia, Canada, Anglia, Suedia, pe baza conferințelor și cursurilor făcute acolo, el a scos o serie de lucrări importante privind categorii abeliene, inele Chow, topologie etală și funcția lui Riemann, funcții recursive etc. Expunerile sale ținute în seminarul Bourbaki au stat la baza unei importante lucrări despre categorii triangulate, apărută în publicațiile IHES.

Prin acestea el și-a atras prețuirea multor matematicieni de seamă ca G. Etoilad, Gr. Moisil, O. Onicescu, Alex. Grothendieck, B. Segal, E. Martinelli, P. Hilton, B. Ekman, P. de Logne, B. Scott și numeroși alții.

Datorită lui, mulți dintre ei ne-au vizitat fără, prețuind dezvoltarea școlii matematice românești, stîrnind entuziasmul, interesul și încrederea tineretului nostru matematic.

Lecțiile sale de algebră pentru studenții de anul I, mobilizatoare prin claritatea, bogăția lor și căldura sa de profesor, rămîn memorabile, ca și cursurile sale pentru studenții anilor mai mari.

Era gata întotdeauna să ne explice pe loc, fără texte ajutătoare, cele mai delicate chestiuni de algebră, topologie, geometrie algebrică, logică matematică, cu o claritate, erudiție, modestie și căldură neîntrecute. În catedră ne-a ajutat adesea cu sfatul său prevenitor, bazat pe o adîncă cunoaștere și pătrundere a problemelor, cu sugestiile sale de mare preț.

A fost insuflețit de înalte aspirații patriotice, iubind intens facultatea și oamenii săi. Îndatoritor cu studenții și colegii, el ne-a mobilizat pe toți pentru îndeplinirea sarcinilor.

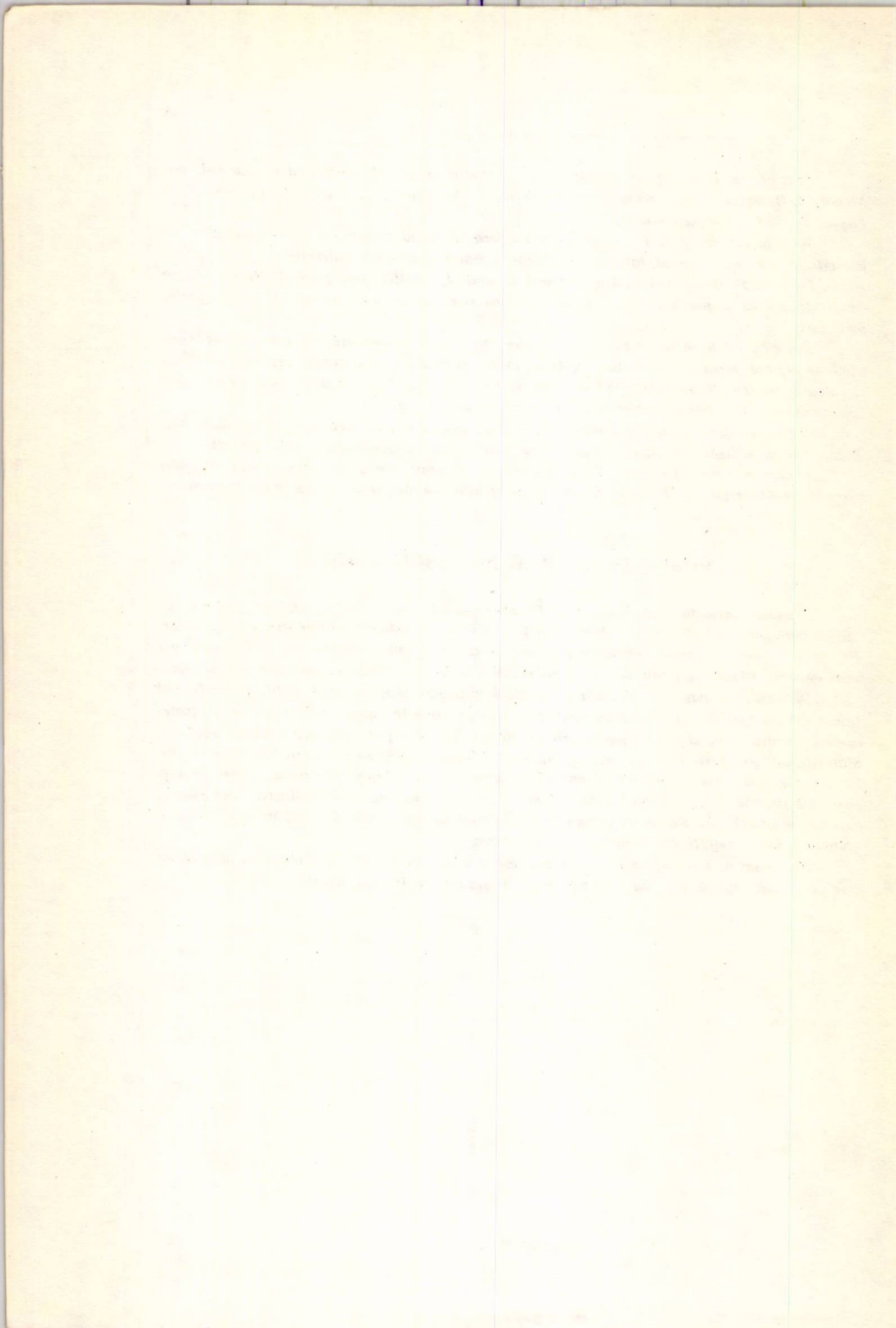
Sîntem îndurerați că Ionel Bucur, colegul și prietenul nostru cu atît de multe și nobile însușiri ne părăsește atît de tînăr. Exemplul său luminos va rămîne mereu viu în inimile noastre.

Cuvîntul Dr. DORIN POPESCU — asistent

Stîngerea prematură din viață a profesorului Ionel Bucur ne-a întristat profund pe toți cei care i-am fost elevi și ne-a folosit din plin marele său talent de om de știință și profesor. Pe noi ne uimea cultura sa bogată ca și ușurința cu care putea îmbina în cadrul aceleiași lecții capitole diferite și profunde de matematică clasică și modernă, abstractă și intuitivă.

Personalitate marcantă în viața matematică internațională, excelent pedagog, profesorul Bucur a încurajat și a ajutat nenumărați tineri să cerceteze în ramuri dificile și noi ale matematicii. Pentru mine și pentru toți tinerii cercetători, dînsul a fost mai mult decît un profesor. Sfaturile sale generoase nu s-au rezumat numai la formarea mea ca matematician ci și ca om. Întotdeauna m-a impresionat umanismul său și arta cu care încerca să rezolve problemele spinuoase ale fiecăruia. Pe absolvenții faultății noastre îi ajuta nu numai în obținerea unor posturi adecvate pregătirii lor, dar și în integrarea lor la noul loc de muncă. La despărțire le spunea: „Nu ne uitați, țineți legătura cu noi, cu facultatea”.

Nu-l vom uita niciodată pe profesorul nostru drag, pe cel care prin opera sa și prin tot ceea ce a făcut pentru noi, va rămîne veșnic nemuritor în inimile noastre.



Abonamentele în țară se pot contracta la oficiile P.T.T.R., agenții poștali și factorii poștali.

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